Chapter 15

Integral of Vector Fields

15.1 Line Integrals

Line integral(Path integral) of a scalar function

Let C be a C^1- curve parameterized by $\mathbf{x}(t) = \mathbf{r}(t) = (x(t), y(t), z(t)) : [a, b] \to C \subset \mathbb{R}^3$.Let P: $a = t_0 < t_1 < \cdots < t_k = b$ be the partition of $[a, b]$ and t_i^* be any point between t_{i-1} and t_i, for $i = 1, \cdots, n$. Then we consider the Riemann sum of a continuous function $f : C \to \mathbb{R}$:

$$\sum_{i=1}^{k} f(\mathbf{x}(t_i^*))\Delta s_i = \sum_{i=1}^{k} f(\mathbf{x}(t_i^*))\|\mathbf{x}(t_i) - \mathbf{x}(t_{i-1})\|.$$

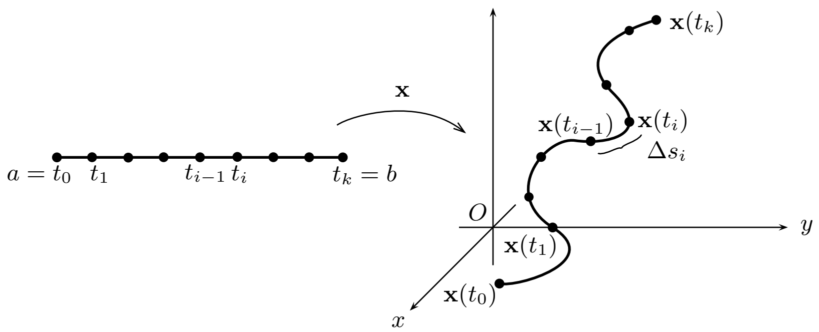


Figure 15.1: Riemann sum over a path

As $\|P\|$ approaches 0 the sum approaches

$$\int_C f(x,y,z)ds = \lim_{k\to\infty} \sum_{i=1}^{k} f(\mathbf{x}(t_i^*))\Delta s_i,$$

where Δs_i is the length of i-th segment of the curve $\mathbf{x}(t)$. (If f denotes the density of a wire lying on the image of C, then the Riemann sum is approximately the total mass of the wire.) Here $s(t)$ is the arc length parameter:

$$s(t) = \int_0^t \|\mathbf{v}(\tau)\| d\tau$$

Definition 15.1.1. If C is a C^1-curve parameterized by $\mathbf{x}(t)$ on $I = [a,b]$ lying in $\mathbb{R}^3$ and f is defined over a region containing the image of $\mathbf{x}$. Then $f \circ \mathbf{x}$ is real valued function defined on I. We define **the line integral - path integral** of f over C as:

$$\int_C f(x,y,z)ds = \int_a^b f(g(t),h(t),k(t))\|\mathbf{v}(t)\|dt = \int_a^b f(\mathbf{x}(t))\|\mathbf{x}'(t)\|\,dt.$$

If $f=1$, then $\int_C ds$ is the length of C.

Example 15.1.2. The scalar function $f(\mathbf{x})$ may represent

(1) electric charge density along the wire represented by $\mathbf{x}(t)$; Then the line integral is total charge along the wire.

(2) density of the wire $\mathbf{x}(t)$. Then the line integral is total mass of the wire

Example 15.1.3. Find path integral of $f(x,y,z) = x^2+y^2+z^2$ over C where

$$\mathbf{x}(t) = (\cos t, \sin t, t), \quad t \in [0, 2\pi].$$

sol. Since $\mathbf{x}'(t) = (-\sin t, \cos t, 1)$, the line integral is

$$\begin{aligned}
\int_C f\,ds &= \int_0^{2\pi} f(\mathbf{x}(t))\|\mathbf{x}'(t)\|\,dt \\
&= \int_0^{2\pi} (\cos^2 t + \sin^2 t + t^2)\|(-\sin t, \cos t, 1)\|\,dt \\
&= \int_0^{2\pi} (1+t^2)\sqrt{2}\,dt \\
&= \sqrt{2}\left(2\pi + 8\pi^3/3\right).
\end{aligned}$$

□

Example 15.1.4 (Tom Sawyer's fence)**.** Find the area of fence built along a path on Tom's yard parameterized by $\mathbf{x} = \mathbf{x}(t) = (30\cos^3 t, 30\sin^3 t), t \in [0, \pi]$ whose height is $f(x, y) = 1 + 3y$. The unit is a foot.

sol. $\mathbf{x}(t) = (30\cos^3 t, 30\sin^3 t)$ for $t \in [0, \pi/2]$. The area of one side is

$$\int_C f(x, y)ds,$$

where $ds = \|\mathbf{x}'(t)\|dt = 90 \sin t \cos t\, dt$. So

$$\begin{aligned}
\int_C f(x, y)ds &= \int_0^{\pi} \left(1 + 10\sin^3 t\right) 90 \sin t \cos t\, dt \\
&= 2\int_0^{\pi/2} \left(1 + 10\sin^3 t\right) 90 \sin t \cos t\, dt \\
&= 180\int_0^{\pi/2} (\sin t + 10\sin^4 t) \cos t\, dt = 225.
\end{aligned}$$

Thus the area of fence(both sides) is 450 (square ft).

□

Line integrals over curves with several components

Let C be an oriented curve which is made up of several oriented curves $C_i, i = 1, 2, \cdots$ (without overlapping except at end points). Then we write

$$C = C_1 + C_2 + \cdots + C_n.$$

Since each C_i can be parameterized separately, we can define the integral over C by

$$\int_C f ds = \int_{C_1} f ds + \int_{C_2} f ds + \cdots + \int_{C_n} f ds.$$

Example 15.1.5. Find path integral of $f(x, y, z) = x^2 + y^2 + z^2$ over C, where

$$C = \{(\cos t, \sin t, t) \colon t \in [0, 2\pi]\} \cup \{(1, 0, t) \colon t \in [0, 2\pi]\}.$$

sol. We write C as the union of C_1 and C_2, where

$$C_1 = \{(\cos t, \sin t, t) \colon t \in [0, 2\pi]\}, \quad C_2 = \{(1, 0, t) \colon t \in [0, 2\pi]\}.$$

We parameterize C_1 and C_2 as follows:

$$\mathbf{x}_1 = (\cos t, \sin t, t), \quad t \in [0,1], \quad \mathbf{x}_2 = (1,0,t), \quad t \in [0, 2\pi].$$

Then

$$\begin{aligned}\int_C f\,ds &= \int_{C_1} f\,ds + \int_{C_2} f\,ds \\ &= \int_0^{2\pi} (1+t^2)\sqrt{2}\,dt + \int_0^{2\pi} (1+t^2)\,dt \\ &= (1+\sqrt{2})\left(2\pi + 8\pi^3/3\right).\end{aligned}$$

□

Mass and Moment of a wire

Imagine coils or springs and wires as masses distributed along smooth curves in space.

When a curve C is parameterized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, a \le t \le b$, the density of wire is $\delta(x(t), y(t), z(t))$.

$M = \int_C \delta\,ds$
$M_{yz} = \int_C x\delta\,ds$
$M_{zx} = \int_C y\delta\,ds$
$M_{xy} = \int_C z\delta\,ds$
$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{zx}}{M}, \bar{z} = \frac{M_{xy}}{M}.$
moment of inertia about the axis and the line L
$I_x = \int_C (y^2+z^2)\delta\,ds,\ I_y = \int_C (x^2+z^2)\delta\,ds,\ I_z = \int_C (x^2+y^2)\delta\,ds,\ I_L = \int_C r^2\delta\,ds.$

Example 15.1.6. Assume a wire is lying on an arc $x = 0, y^2 + z^2 = 1,\ z \ge 0$ of yz-plane. Find the center of mass of the mass if the density is given by $\delta(x,y,z) = 2 - z$.

sol. We know $\bar{x} = \bar{y} = 0$ since the arc lies symmetrically about z axis. To find $\bar{z}$ we parameterize the semi circle

$$y = \cos t,\ z = \sin t, \quad 0 \le t \le \pi.$$

$$|\mathbf{x}'(t)| = \|(-\sin t, \cos t)\| = 1$$

so $ds = dt$.

$$\begin{aligned}
M &= \int_C \delta ds = \int_0^\pi (2 - \sin t)dt = 2\pi - 2. \\
M_{xy} &= \int_C z\delta ds = \int_0^\pi (\sin t)(2 - \sin t)dt \\
&= \int_0^\pi (2\sin t - \sin^2 t)dt = \frac{8-\pi}{2}. \\
\bar{z} &= \frac{M_{xy}}{M} = \frac{8-\pi}{2}\frac{1}{2\pi - 2} = \frac{8-\pi}{4\pi - 4}.
\end{aligned}$$

□

15.2 Vector fields and Line integral: Work, Circulation and Flux

Vector fields

Imagine a lot of arrows (vectors) spread over a (corn) field.

Definition 15.2.1. A **vector field** is a vector valued function defined on a domain:

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}.$$

The field is **continuous** (resp. **differentiable**) if its components are continuous (resp. **differentiable**).

As an example, you can think of velocity vector of a moving fluid (river). At each point of the domain (river), there is an associated velocity vector denoting the fluid flow.

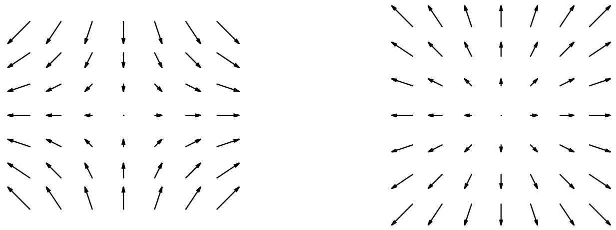
Gradient fields and potentials

Given real C^1- function $f(x_1, x_2, \ldots, x_n)$, we define the **gradient field** by

$$\nabla f := (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}).$$

If a vector field $\mathbf{F}$ is given by

$$\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x}),$$



$F(x, y) = (x, -y)$ $\qquad$ $F(x, y) = (x, y)$

Figure 15.2: Vector fields

for some scalar function f, then f is called the **potential function**.

Line Integrals of Vector Fields

As an example, consider the work done by a force field. Suppose a particle moves along a curve $\mathbf{x}$ while acted upon by a force $\mathbf{F}$. If a portion of $\mathbf{x}$ is a line segment given by the vector $\Delta\mathbf{x}$ and $\mathbf{F}$ is constant force, then the work done on the particle along $\Delta\mathbf{x}$ is, by definition

$$\boxed{\text{Work } = \mathbf{F} \cdot \Delta\mathbf{x} = \text{ magnitude of force } \times \text{ displacement in the direction of force.}}$$

If the path is a curve, we break the curve into small pieces and add the work

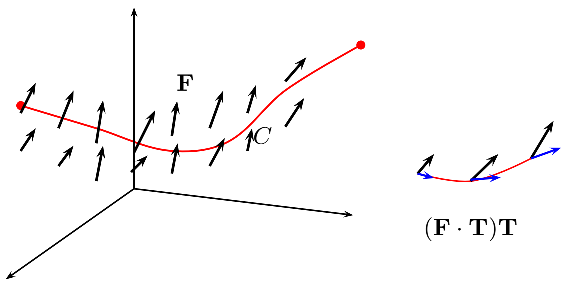


Figure 15.3: Line integral of vector fields is integral of tangential projection

done on each piece then take the limit. The the work done on the i-th piece is $\approx \mathbf{F}(\mathbf{x}(t_i^*)) \cdot \Delta\mathbf{x}_i$. So the work is defined by

$$\sum_{i=0}^{n-1} \mathbf{F}(\mathbf{x}(t_i^*)) \cdot \Delta\mathbf{x}_i = \sum_{i=0}^{n-1} \mathbf{F}(\mathbf{x}(t_i)) \cdot [\mathbf{x}(t_i + \Delta t) - \mathbf{x}(t_i)].$$

Taking the limit

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=0}^{n-1}\mathbf{F}(\mathbf{x}(t_i^*))\cdot\Delta\mathbf{x}_i &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\mathbf{F}(\mathbf{x}(t_i))\cdot\frac{\Delta\mathbf{x}_i}{\Delta t}\Delta t\\
&= \int_a^b \mathbf{F}(\mathbf{x}(t))\cdot\mathbf{x}'(t)dt.
\end{aligned}$$

This line integral can be interpreted as integral of the tangential component of $\mathbf{F}\cdot\mathbf{T}$ along the curve as follows: For $\mathbf{x}'(t)\neq 0$, we see the vector $\mathbf{T}(t) = \mathbf{x}'(t)/\|\mathbf{x}'(t)\|$ is the unit tangent vector. Hence

$$\begin{aligned}
\int_a^b \mathbf{F}(\mathbf{x}(t))\cdot\mathbf{x}'(t)dt &= \int_a^b \left[\mathbf{F}(\mathbf{x}(t))\cdot\frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}\right]\|\mathbf{x}'(t)\|dt\\
&= \int_a^b [\mathbf{F}(\mathbf{x}(t))\cdot\mathbf{T}(t)]\,\|\mathbf{x}'(t)\|dt\\
&= \int_C (\mathbf{F}\cdot\mathbf{T})\,ds \equiv \int_C \mathbf{F}\cdot\, d\mathbf{x}.
\end{aligned}$$

Note also that

$$\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = \frac{d\mathbf{x}(t)/dt}{\|\mathbf{x}'(t)\|} = \frac{d\mathbf{x}}{ds}.$$

Definition 15.2.2. Let $\mathbf{F}$ be a continuous vector field on $\mathbb{R}^3$ that is defined on a set containing the image of C^1 - curve $\mathbf{x}:[a,b]\to\mathbb{R}^3$. Define the **(vector) line integral** of $\mathbf{F}$ along the curve as

$$\int_C (\mathbf{F}\cdot\mathbf{T})\,ds = \int_C \mathbf{F}\cdot\frac{d\mathbf{x}}{ds}ds = \int_C \mathbf{F}\cdot d\mathbf{x}.$$

So the line integral of a vector field is the **path integral(scalar line integral) of the tangential component** $\mathbf{F}\cdot\mathbf{T}$ along the curve.

Example 15.2.3. Suppose $\mathbf{F}(x,y,z) = x^3\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the curve C is a circle given by $x = 0, y^2+z^2 = a^2$. Compute $\int_C \mathbf{F}\cdot\, d\mathbf{x}$.

sol. We parameterize the circle $\mathbf{x}(t) = (x(t), y(t), z(t))$ by

$$x = 0,\ y = a\cos t,\ z = a\sin t,\quad 0\le t\le 2\pi.$$

$$\mathbf{x}'(t) = (0, -a\sin t, a\cos t).$$

Since $\mathbf{F}(\mathbf{x}(t))\cdot\mathbf{x}'(t) = 0$, the work must be zero. You can verify by finding the value.

$\square$

Line integral with resp. to dx, dy or dz

Suppose the vector field

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

is given and

$$\mathbf{r}(t) \equiv \mathbf{x}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},\ \ a \le t \le b$$

is a smooth curve.Then recalling $\mathbf{r}'(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$, we see

$$\int_a^b \mathbf{F}(\mathbf{r}(t))\cdot d\mathbf{r} = \int_a^b (M, N, P)\cdot(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})dt = \int_C M dx + N dy + P dz. \quad (15.1)$$

Thus the lintegral of a vector field $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is written in the form of (15.1).

Flow integrals and circulation of velocity fields

Definition 15.2.4. If C is a smooth curve in the domain of a continuous vector field $\mathbf{F}$ and $\mathbf{T}$ is unit tangent vector on C, the **flow** of $\mathbf{F}$ along C from $A = \mathbf{x}(a)$ to $B = \mathbf{x}(b)$ is

$$\int_C \mathbf{F}\cdot\mathbf{T}\, ds.$$

If the curve is closed, i.e, $A = B$, then the flow is called the **circulation** of $\mathbf{F}$ along C.

Example 15.2.5. Let $\mathbf{F}(x, y, z) = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$. Find the flow of $\mathbf{F}$ along the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k},\ 0 \le t \le \pi/2$.

sol.

$$\frac{d\mathbf{r}}{dt} = (-\sin t, \cos t, 1).$$

$$\begin{aligned}\int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} &= \int_0^{\pi/2} (-\sin t\cos t + t\cos t + \sin t)\, dt \\ &= \left[\frac{\cos^2 t}{2} + t\sin t\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - \frac{1}{2}.\end{aligned}$$

□

Example 15.2.6. Find the circulation of $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + x\mathbf{j}$ along the circle $x^2 + y^2 = 1$.

sol. Parameterize the unit circle by $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ $(0 \le t \le 2\pi)$. Then on the circle $\mathbf{F} = (\cos t - \sin t)\mathbf{i} + \cos t\mathbf{j}$ and

$$\frac{d\mathbf{r}}{dt} = (-\sin t, \cos t, 0), \ \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + 1.$$

The circulation is

$$\begin{aligned} \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt &= \int_0^{2\pi} (1 - \sin t \cos t)\, dt \\ &= \left[t - \frac{\sin^2 t}{2} \right]_0^{2\pi} = 2\pi. \end{aligned}$$

□

Flux across a simple closed plane curve

Definition 15.2.7. If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F}$ and $\mathbf{n}$ is unit outward normal vector on C, the **flux** of $\mathbf{F}$ across C is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds.$$

Calculating flux across a simple closed plane curve:

Let $(x(t), y(t))$ be a parametrization of C and $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$. Then unit normal vector is (Fig. 15.4)

$$\mathbf{n} = \mathbf{t} \times \mathbf{k} = \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} \right) \times \mathbf{k} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

$$\mathbf{F} \cdot \mathbf{n} = M(x, y)\frac{dy}{ds} - N(x, y)\frac{dx}{ds}.$$

Hence the flux is

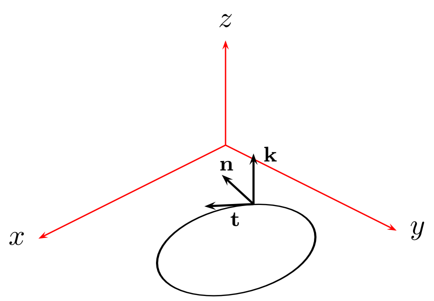


Figure 15.4: Outward normal $\mathbf{n} = \mathbf{t} \times \mathbf{k}$ directs the rhs of a walking man

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds} \right) ds \tag{15.2}$$

$$= \oint_C M dy - N dx. \tag{15.3}$$

Example 15.2.8. Find the flux of $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + x\mathbf{j}$ along the circle $x^2 + y^2 = 1$. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ $(0 \le t \le 2\pi)$.

sol. We see $\frac{d\mathbf{r}}{dt} = (-\sin t, \cos t)$. Hence

$$dy = \cos t, \quad dx = \sin t.$$

Since

$$M = x - y = \cos t - \sin t, \; N = x = \cos t$$

we see the flux is

$$\begin{aligned}\int_C M dy - N dx &= \int_0^{2\pi} (\cos^2 t - \sin t \cos t + \sin t \cos t)\, dt \\ &= \int_0^{2\pi} \cos^2 t\, dt = \int_0^{2\pi} \frac{1 + \cos 2t}{2} dt \\ &= \left[\frac{t}{2} + \frac{\sin 2t}{4} \right]_0^{2\pi} = \pi.\end{aligned}$$

□

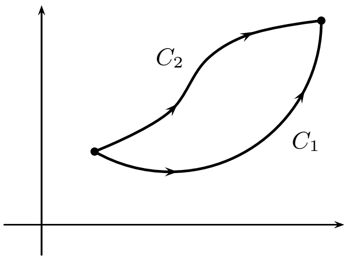


Figure 15.5: Two curves having the same end points

15.3 Path independence, conservative vector fields

Definition 15.3.1. A line integral a vector field $\mathbf{F}$ is called **path independent** if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \tag{15.4}$$

for any two oriented curves C_1, C_2 lying in the domain of $\mathbf{F}$ having same end points. The field is called **conservative**.

A vector field $\mathbf{F}$ is called a **gradient vector field** if $\mathbf{F} = \nabla f$ for some real valued function f. Thus

$$\mathbf{F} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

The function f is called a **potential** of $\mathbf{F}$.

Example 15.3.2. A gravitational force field has the potential function $f = \frac{GmM}{r}$ $(\mathbf{r} = (x, y, z),\ r = \sqrt{x^2 + y^2 + z^2})$.

$$\mathbf{F} = -\frac{GmM}{r^3}\mathbf{r} = \nabla f.$$

sol. We take derivative of $r^2 = x^2 + y^2 + z^2$, i.e., $2r\frac{\partial r}{\partial x} = 2x, 2r\frac{\partial r}{\partial y} = 2y, 2r\frac{\partial r}{\partial z} = 2z$. Thus

$$\nabla f = -\frac{GmM}{r^2}\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = -\frac{GmM}{r^3}\mathbf{r}.$$

□

Theorem 15.3.3. *Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is class C^1 and $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is smooth curve C and $\mathbf{F}$ is a continuous gradient field such that $\mathbf{F} = \nabla f$. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

In other words, the gradient field is conservative.

Proof. By the chain rule, we get

$$(f \circ \mathbf{r})'(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

So

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

So the line integral is independent of parametrization. □

Theorem 15.3.4. ***(Conservative Field)*** *Let $\mathbf{F}$ be a $\mathcal{C}^1$-vector field on an open connected region D in $\mathbb{R}^3$. Then $\mathbf{F}$ is conservative if and only if*

$$\mathbf{F} = \nabla f$$

for some f.

Proof. We have already seen that if $\mathbf{F} = \nabla f$, then it is conservative. We thus need to prove its converse.

Let A and B are any two points in the region D. Assume

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{x} = \int_{C_2} \mathbf{F} \cdot d\mathbf{x}$$

for any two curves C_1 and C_2 connecting A and B. Let $\mathbf{F} = (F_1, F_2, F_3)$, $A = (x_0, y_0, z_0)$ and $B = (x, y, z)$ and let C be any curve connecting A to B. Define

$$f(x, y, z) = \int_C \mathbf{F} \cdot d\mathbf{x} = \int_C F_1 dx + F_2 dy + F_3 dz.$$

Here f is well-defined, since it is defined independent of the choice of C. In particular, we choose C consisting of a curve and union of edges of certain rectangular box. Consider a small box with opposite vertices (x_0, y_0, z_0) and (x_1, y_1, z_1) contained domain (fig 15.6).

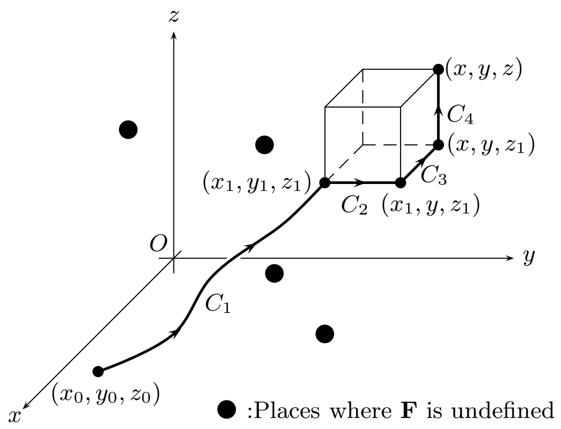


Figure 15.6: $C = C_1 + C_2 + C_3 + C_4$ connects (x_0, y_0, z_0) and (x, y, z)

Choose $C = C_1 + C_2 + C_3 + C_4$, where C_1 is a curve connecting (x_0, y_0, z_0) to (x_1, y_1, z_1) and C_2, C_3, and C_4 are edges of the box. Then

$$\begin{aligned} f(x, y, z) &= \int_C \mathbf{F} \cdot d\mathbf{x} \\ &= \int_{C_1} \mathbf{F} \cdot d\mathbf{x} + \int_{C_2} \mathbf{F} \cdot d\mathbf{x} + \int_{C_3} \mathbf{F} \cdot d\mathbf{x} + \int_{C_4} \mathbf{F} \cdot d\mathbf{x} \\ &= \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{y_1}^{y} F_2(x_1, t, z_1)dt + \int_{x_1}^{x} F_1(t, y, z_1)dt + \int_{z_1}^{z} F_3(x, y, t)dt. \end{aligned}$$

From this we see $\partial f/\partial z = F_3$. Similarly, by choosing different path(i.e, choosing a path whose $\underline{\text{last path is along } x\text{-direction}}$) we have

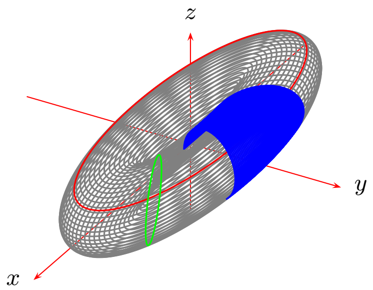
$$f(x, y, z) = \int_{C_1} \mathbf{F} \cdot d\mathbf{x} + \int_{y_1}^{y} F_2(x_1, t, z_1)dt + \int_{z_1}^{z} F_3(x_1, y, t)dt + \int_{x_1}^{x} F_1(t, y, z)dt.$$

So $\partial f/\partial x = F_1$. Similarly, by choosing appropriate curve, we can show $\partial f/\partial y = F_2$. Thus $\mathbf{F} = \nabla f$.

□

Definition 15.3.5. A region R in $\mathbb{R}^2$ or $\mathbb{R}^3$ is called **simply connected** if every closed curve C in R can be continuously shrunk to a point (contractible) while remaining in R throughout the deformation.

Example 15.3.6. The 2 dimensional plane with a disk removed is not **simply connected**, but 3 dimensional space is still **simply connected** if a finite number of disks are removed. The 3 dimensional space is not **simply con-**



Red curve is not contractible

Figure 15.7: Solid of torus is not simply connected

nected if a line(or an infinite cylinder) is removed.

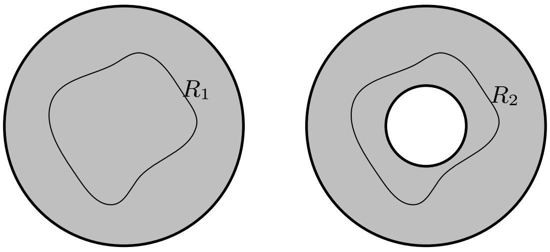


Figure 15.8: 'Simply connected' and 'not simply connected' region

Example 15.3.7. Let $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ and consider two paths C_1 and C_2 connecting $(0,0)$ and $(1,1)$. We compare $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. These curves may be parameterized as

$$C_1 : \begin{cases} x = t \\ y = t \end{cases} \quad (0 \le t \le 1) \text{ and } C_2 : \begin{cases} x = t \\ y = t^2 \end{cases} \quad (0 \le t \le 1).$$

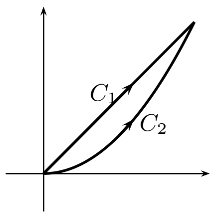


Figure 15.9: Two paths connecting $(0,0)$ and $(1,1)$

Curl of a vector field in $\mathbb{R}^3$

If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} = (F_1, F_2, F_3)$, then $\nabla \times \mathbf{F}$ ($\equiv \mathbf{curl}\,\mathbf{F}$) is defined as

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \Big(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix} = \Big(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\Big)\mathbf{i} + \Big(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\Big)\mathbf{j} + \Big(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\Big)\mathbf{k}.$$

Component test for conservative field

If a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative on a simply connected domain, then by above Theorem, there exists some function f s.t.

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Hence we can check the following holds: (by taking the derivative)

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \text{ and } \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{15.5}$$

Theorem 15.3.8. *Let $\mathbf{F}$ be a $\mathcal{C}^1$-vector field on an open simply connected region D in $\mathbb{R}^3$. Then $\mathbf{F}$ is conservative if and only if (15.5) holds (in other words, $\nabla \times \mathbf{F} = 0$).*

Proof. If $\mathbf{F}$ is conservative, we have just seen that $\mathbf{F} = \nabla f$ for some f. Then by checking, we can easily see (15.5) holds. To show the converse holds, we need Stokes' theorem(Later) □

Example 15.3.9. Show that the vector field is conservative and find its potential.

$$\mathbf{F}(x,y,z) = (e^x \sin y - yz)\mathbf{i} + (e^x \cos y - xz)\mathbf{j} + (z - xy)\mathbf{k}.$$

sol. One can check (15.5) or check if the curl $\mathbf{F}$ is zero:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y - yz & e^x \cos y - xz & z - xy \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(z - xy) - \frac{\partial}{\partial z}(e^x \cos y - xz)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(e^x \sin y - yz) - \frac{\partial}{\partial x}(z - xy)\right)\mathbf{j} \\ &+ \left(\frac{\partial}{\partial x}(e^x \cos y - xz) - \frac{\partial}{\partial y}(e^x \sin y - yz)\right)\mathbf{k} = \mathbf{0}.\end{aligned}$$

So the condition (15.5) holds. To find a potential we need to find and f satisfying

$$\frac{\partial f}{\partial x} = e^x \sin y - yz, \quad \frac{\partial f}{\partial y} = e^x \cos y - xz, \quad \frac{\partial f}{\partial z} = z - xy. \tag{15.6}$$

Thus we proceed as follows: First integrate w.r.t x.

(1) $f(x,y,z) = \int (e^x \sin y - yz)dx = e^x \sin y - xyz + g(y,z)$ for some $g(y,z)$.

(2) $\frac{\partial f}{\partial y} = e^x \cos y - xz + \frac{\partial g}{\partial y} = e^x \cos y - xz$. Thus $g(y,z)$ is a function of z only, thus $g = g(z)$. Taking derivative of f w.r.t z, we have

(3) $\frac{\partial f}{\partial z} = -xy + g'(z) = z - xy$. Thus $g(z) = \frac{1}{2}z^2 + C$.

(4) Hence $f(x,y,z) = e^x \sin y - xyz + \frac{1}{2}z^2 + C$.

□

For a non simply connected region, curl free does not imply conservativeness.

Example 15.3.10. Consider the vector field

$$\mathbf{F}(x,y) = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} + 0\mathbf{k}.$$

This field satisfies $\nabla \times \mathbf{F} = 0$ except the origin. But it is not conservative. Explain why.

sol. Need to check

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \text{ and } \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{15.7}$$

In this field $P = 0$ and no z variable. Hence to check the condition it is sufficient to check

$$\frac{\partial N}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial M}{\partial y}.$$

We evaluate this integral along the circle : $\mathbf{r}(t) = (\cos t, \sin t), 0 \le t \le 2\pi$.

$$\mathbf{F} = -\sin t\mathbf{i} + \cos t\mathbf{j}, \quad \mathbf{r}'(t) = (-\sin t, \cos t).$$

Hence

$$\begin{aligned}\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{x} &= \int_0^{2\pi} (-\sin t\mathbf{i} + \cos t\mathbf{j}) \cdot (-\sin t, \cos t)\, dt \\ &= \int_0^{2\pi} 1\, dt = 2\pi \ne 0.\end{aligned}$$

Hence this field is not conservative. But this does not contradict Theorem 15.3.4 because the domain in this case is not simply connected.

□

Exact differential form

Consider a vector field $\mathbf{F} = (F_1, F_2, F_3)$ and a parameterized curve $\mathbf{x}(t) = (x(t), y(t), z(t))$. Since $(dx, dy, dz) = (x'(t), y'(t), z'(t))dt$, we can write the line integral as

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (F_1, F_2, F_3) \cdot (x'(t), y'(t), z'(t))dt \\ &= \int_C F_1 dx + F_2 dy + F_3 dz.\end{aligned}$$

The expression $F_1 dx + F_2 dy + F_3 dz$ is called a **differential form**.

Definition 15.3.11. A differential form is said to be **exact** if it has the form

$$Mdx + Ndy + Pdz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \equiv df = \nabla f \cdot d\mathbf{x}.$$

for some scalar function f.

Component test for exactness

The differential form is exact if and only if (following Theorem 15.3.4)

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \text{ and } \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{15.8}$$

This is a consequence of Theorem 15.3.4 for conservative field.

Example 15.3.12. Find the potential of the vector field if it is conservative.

$$\mathbf{F}(x, y) = (2xy + \cos 2y)\mathbf{i} + (x^2 - 2x\sin 2y)\mathbf{j}.$$

sol.

First we check that $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$. Hence it is conservative. Let f be the potential function. Then it satisfies $\nabla f = \mathbf{F}$, i.e.,

$$\frac{\partial f}{\partial x} = 2xy + \cos 2y, \quad \frac{\partial f}{\partial y} = x^2 - 2x\sin 2y. \tag{15.9}$$

Thus we proceed as follows:

(1) Integrate: $f(x, y) = \int \frac{\partial f}{\partial x}\, dx = \int 2xy + \cos 2y\, dx = x^2 y + x\cos 2y + g(y)$

(2) Set $\frac{\partial f}{\partial y} = x^2 - 2x\sin 2y + g'(y)$

(3) Show $g(x, y) = C$.

Thus we see $f(x, y) = x^2 - 2x\sin 2y + C$.

□

Example 15.3.13. Show the form $ydx + xdy + 4dz$ is exact and evaluate the integral

$$\int_C ydx + xdy + 4dz.$$

Here, C is a curve having A and B as the beginning point and end point.

sol.

First we check (15.8) or

$$\textbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 4 \end{vmatrix} = 0.$$

Hence it is conservative. Let f be the potential function. Then it satisfies $\nabla f = \mathbf{F}$, i.e.,

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = 4. \tag{15.10}$$

Thus we proceed as follows:

(1) Integrate: $f(x, y) = \int y dx = xy + g(y, z)$

(2) Set $\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x \Rightarrow g = h(z)$

(3) $\frac{\partial f}{\partial z} = 4 \Rightarrow h = 4z + C$

Thus $f = xy + 4z + C$ and $\int_C y dx + x dy + 4dz = f(B) - f(A)$.

□

15.4 Green's Theorem in the plane

Circulation and flux

We first consider two important concept for a fluid (flow) related to a closed region in 2D. (can be extended to 3D later) Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be the vector field representing the velocity of some fluid. Then

(1) The **circulation rate** measures the spin of the fluid around a closed curve, which is given $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C M dx + N dy$.

(2) The **flux rate** measures the rate at which the fluid leaves out of the closed curve, which is given $\oint_C \mathbf{F} \cdot \mathbf{n} ds = \oint_C M dy - N dx$.

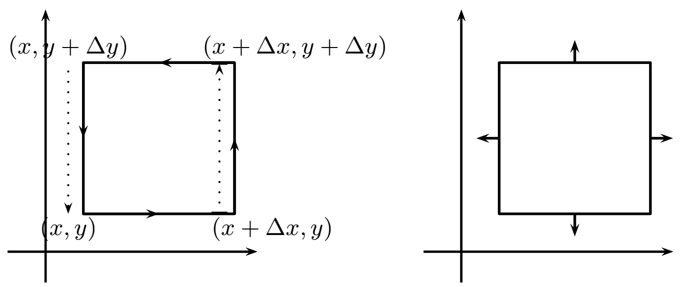


Figure 15.10: Circulation and flux

1. **Circulation** along the closed curve C is $\oint_C Mdx + Ndy$. We compute it along the horizontal lines and vertical lines. (Fig. 15.10)

$$\left\{\begin{array}{ll} \text{top} & \int -M(x, y+\Delta y)dx \\ \text{bottom} & \int M(x,y)dx \end{array}\right\} \quad => \quad \int_x^{x+\Delta x}\int_y^{y+\Delta y} -\frac{\partial M}{\partial y}dydx$$

$$\left\{\begin{array}{ll} \text{right} & \int N(x+\Delta x, y)dy \\ \text{left} & \int -N(x,y)dy \end{array}\right\} \quad => \quad \int_y^{y+\Delta y}\int_x^{x+\Delta x} \frac{\partial N}{\partial x}dxdy$$

Adding these, we get

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy. \tag{15.11}$$

Dividing it by the area $A(R)$ and taking the limit as the area approaches zero we get $\boxed{\text{the } \textbf{circulation density } \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \text{ at each point.}}$

2. **Flux** across the closed curve C is $\oint_C Mdy - Ndx$. Again we compute it along the horizontal lines and vertical lines. (Fig. 15.10)

$$\left\{\begin{array}{ll} \text{top} & \int N(x, y+\Delta y)dx \\ \text{bottom} & \int -N(x,y)dx \end{array}\right\} \quad => \quad \int_x^{x+\Delta x}\int_y^{y+\Delta y} \frac{\partial N}{\partial y}dydx$$

$$\left\{\begin{array}{ll} \text{right} & \int M(x+\Delta x, y)dy \\ \text{left} & \int -M(x,y)dy \end{array}\right\} \quad => \quad \int_y^{y+\Delta y}\int_x^{x+\Delta x} \frac{\partial M}{\partial x}dxdy$$

Adding these, we get

$$\oint_C Mdy - Ndx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy. \tag{15.12}$$

Dividing it by the area $A(R)$ and taking the limit as the area approaches zero we get $\boxed{\textbf{the flux density } \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \equiv \operatorname{div}\mathbf{F} \text{ at each point.}}$ [1]

These are two versions of Green's theorem on rectangular regiosn. They hold much more general regions.

Remark 15.4.1. The second version of Green's theorm, i.e., the relation (15.12) follows from the circulation version (15.11) by replacing N by M and M by$-N$.

[1]For 3D case, $\operatorname{div}\mathbf{F} \equiv \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$ for $\mathbf{F} = (M, N, P)$.

Relation with 3D curl

If $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is two dimensional vector field, then it can be considered as a three dimensional vector field as $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} + 0 \cdot \mathbf{k}$. The **curl F** can be computed :

$$\begin{aligned}\textbf{curl } \mathbf{F} &= \Big(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\mathbf{k} \\ &= \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\mathbf{k}.\end{aligned}$$

Definition 15.4.2. The **circulation density** of $\mathbf{F}$ is the expression $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$, also called the $\mathbf{k}$ - component of the curl denoted by $(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}$.

Physical meaning:

(1) The integral of a circulation around a closed curve is the same as the integral of the curl of $\mathbf{F}$ on the region enclosed by the curve.

(2) Normal component of **curl F** is the rate of **rotation** along the plane.

Green's Theorem

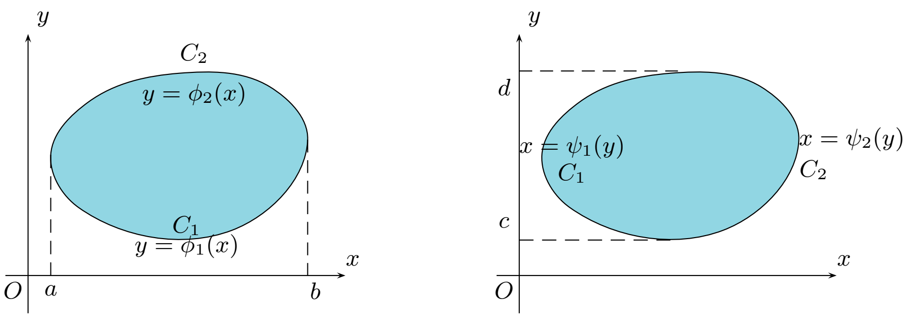


Figure 15.11: As type 1 region and boundary

Theorem 15.4.3. ***(Green's theorem: Circulation-Curl form)*** *Let D be a closed bounded, region in $\mathbb{R}^2$ with boundary $C = \partial D$ with positive orientation. (The region D is on the left side as one traverses C.) Here ∂D denotes the boundary of D and $\oint_{\partial D}$ means that the integral is defined on a closed curve.*

Suppose $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ *be a vector field of class* $\mathcal{C}^1$. *Then*

$$\oint_{\partial D} \mathbf{F}\cdot\mathbf{T}\,ds = \oint_{\partial D} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

The integral of the **circulation** *around a* ∂D *is the integral of* $\mathbf{curl}\,\mathbf{F}\cdot\mathbf{k}$ *on* D.

Proof. Assume D is a region of type 1 given as follows:

$$D = \{(x,y)|\ a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\}.$$

We decompose the boundary of D as $\partial D = C_1^+ + C_2^-$(fig 15.11). Using the Fubini's theorem, we can evaluate the double integral as an iterated integral

$$\begin{aligned}\iint_D -\frac{\partial M(x,y)}{\partial y} dxdy &= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} -\frac{\partial M(x,y)}{\partial y} dydx \\ &= \int_a^b [M(x,\phi_1(x)) - M(x,\phi_2(x))]dx.\end{aligned}$$

On the other hand, C_1^+ can be parameterized as $x \to (x,\phi_1(x)), a \le x \le b$ and C_2^+ can be parameterized as $x \to (x,\phi_2(x)), a \le x \le b$. Hence

$$\int_a^b M(x,\phi_i(x))dx = \int_{C_i^+} M(x,y)dx, \quad i = 1,2.$$

By reversing orientations

$$-\int_a^b M(x,\phi_2(x))dx = \int_{C_2^-} M(x,y)dx.$$

Hence

$$\iint_D -\frac{\partial M}{\partial y}\,dxdy = \int_{C_1^+} M\,dx + \int_{C_2^-} M\,dx = \int_{\partial D} M\,dx.$$

Similarly if D is a region of type 2, one can show that

$$\iint_D \frac{\partial N}{\partial x} dxdy = \int_{C_1^+} Ndy + \int_{C_2^-} Ndy = \int_{\partial D} N\,dy.$$

Here C_1 and C_2 are the curves defined by $x = \psi_1(y)$ and $x = \psi_2(y)$ for $c \le y \le d$. The proof is completed. ☐

Theorem 15.4.4. ***(Green's theorem: Flux-Divergence form)*** *Let D be a closed bounded, region in $\mathbb{R}^2$ with boundary $C = \partial D$ with positive orientation.*

Suppose $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ be a vector field of class $\mathcal{C}^1$. Then

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds = \oint_{\partial D} M\, dy - N\, dx = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy.$$

The integral of the outward **flux** *around a ∂D = the integral of div* $\mathbf{F}$ *on D.*

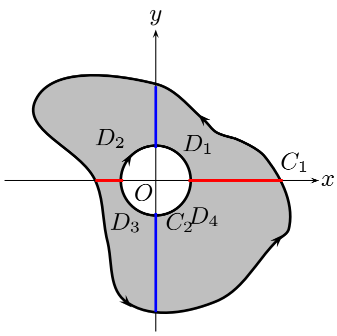


Figure 15.12: Apply Green's theorem to each of the regions

Interior of a region

Let D be a region in $\mathbb{R}^2$. We denote its boundary by ∂D and assume its orientation is given in the counterclockwise direction, i.e, when one walks along the boundary, the region on his left is assumed to be **interior.** (Fig. 15.12)

Generalizing Green's theorem-may skip

In fact, Green's theorem holds for more general region. For example, Green's theorem can be used for a region with a hole. One cuts the region so that each region is type 3.

Theorem 15.4.5. ***(Green's theorem for general region)*** *Let D be a region which can be divided into a several pieces of regions where Green's theorem apply, and let ∂D be the boundary. Suppose M and $N : D \to \mathbb{R}$ are $\mathcal{C}^1$ func-*

tions, then

$$\int_{\partial D} Mdx + Ndy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

Proof. Assume D is the union of type 3 regions D_i, $i = 1, 2, \ldots, n$ whose boundary ∂D is the sum of ∂D_i, $i = 1, 2, \ldots, n$. In other words,

$$D = \sum_{i=1}^{n} D_i \ , \qquad \partial D = \sum_{i=1}^{n} \partial D_i.$$

So

$$\int_{\partial D} Mdx + Ndy = \sum_{i=1}^{n} \int_{\partial D_i} Mdx + Ndy.$$

and

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \sum_{i=1}^{n} \iint_{D_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

Since each D_i is type 3, we can apply Theorem 15.4.3 to have

$$\int_{\partial D_i} Mdx = \iint_{D_i} -\frac{\partial M}{\partial y} dxdy$$

and

$$\int_{\partial D_i} Ndy = \iint_{D_i} \frac{\partial N}{\partial x} dxdy.$$

We add all these terms to get the result. □

Example 15.4.6. Verify Green's theorem for

$$M(x, y) = \frac{-y}{x^2 + y^2}, \quad N(x, y) = \frac{x}{x^2 + y^2}$$

on $D = \{(x, y) |\ h^2 \le x^2 + y^2 \le 1\}$, $0 < h < 1$.

sol. The boundary of D consists of two circles.

$$\begin{aligned} C_1 &: x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le 2\pi \\ C_h &: x = h\cos t, \qquad y = h\sin t, \qquad 0 \le t \le 2\pi. \end{aligned}$$

In the curve $\partial D = C_h \cup C_1$, C_1 is oriented counterclockwise while C_h is oriented clockwise. Since M, N are class $\mathcal{C}^1$ in the annuls D, we can use

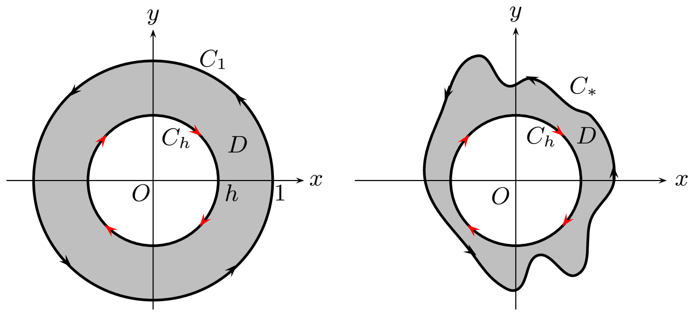


Figure 15.13: Domains for Example 15.4.6 and Example 15.4.7

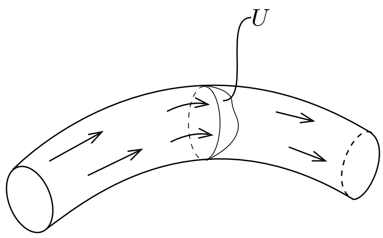


Figure 15.14: Flow across surface U

Green's theorem. Since

$$\frac{\partial M}{\partial y} = \frac{(x^2+y^2)(-1)+2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial N}{\partial x}$$

we have

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_D 0\ dxdy = 0.$$

On the other hand,

$$\begin{aligned}\int_{\partial D} Mdx + Ndy &= \int_{C_1} \frac{xdy - ydx}{x^2+y^2} + \int_{C_h} \frac{xdy-ydx}{x^2+y^2} \\ &= \int_0^{2\pi} (\cos^2 t + \sin^2 t)dt + \int_{2\pi}^0 \frac{h^2(\cos^2 t + \sin^2 t)}{h^2} dt \\ &= 2\pi - 2\pi = 0.\end{aligned}$$

Hence

$$\int_{\partial D} Mdx + Ndy = 0 = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

□

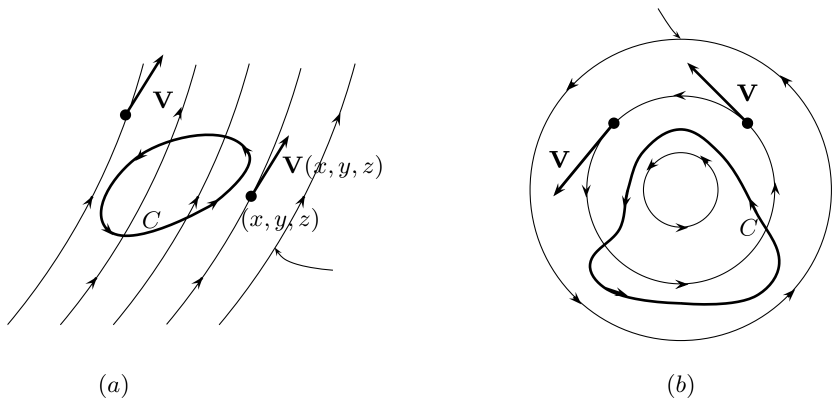


Figure 15.15: (a) circulation zero; (b) nonzero circulation

Example 15.4.7. Evaluate $\int_{C_*} \frac{xdy-ydx}{x^2+y^2}$ where C_* is any closed curve around the origin.

sol. Since the integrand is not continuous at $(0,0)$, we cannot use Green's theorem on the interior of C_*. But if we remove a small circle of radius h around the origin, we can use the Green's theorem on the region bounded by C_* and C_h (Fig 15.13) as in the previous example to see

$$\int_{C_*} Mdx + Ndy = -\int_{C_h} Mdx + Ndy.$$

Now the integral $-\int_{C_h}(Mdx + Ndy)$ can be computed by polar coordinate: From

$$\begin{aligned} x &= h\cos\theta, \qquad y = h\sin\theta, \\ dx &= -h\sin\theta d\theta, \\ dy &= h\cos\theta d\theta, \end{aligned}$$

we see

$$\frac{xdy - ydx}{x^2+y^2} = \frac{h^2(\cos^2\theta + \sin^2\theta)}{h^2}d\theta = d\theta.$$

Hence

$$\int_{C_*} \frac{xdy - ydx}{x^2+y^2} = 2\pi.$$

□

Area

Theorem 15.4.8. *If C is a simple closed curve bounding a region D, then the area A is*

$$A = \frac{1}{2}\int_{\partial D} x dy - y dx.$$

Proof. Let $M(x,y) = -y, N(x,y) = x$. Then

$$\begin{aligned}\frac{1}{2}\int_{\partial D} x dy - y dx &= \frac{1}{2}\iint_D \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dx dy \\ &= \frac{1}{2}\iint_D (1+1) dx dy = \iint_D dx dy = A.\end{aligned}$$

□

Example 15.4.9. Find the area of the region enclosed by $x^{2/3} + y^{2/3} = a^{2/3}$.

sol. Let $x = a\cos^3\theta$, $y = a\sin^3\theta$, $(0 \le \theta \le 2\pi)$. Then

$$\begin{aligned}A &= \frac{1}{2}\int_{\partial D} x dy - y dx \\ &= \frac{1}{2}\int_0^{2\pi} [(a\cos^3\theta)(3a\sin^2\theta\cos\theta) - (a\sin^3\theta)(-3a\cos^2\theta\sin\theta)] d\theta \\ &= \frac{3}{2}a^2\int_0^{2\pi} (\sin^2\theta\cos^4\theta + \cos^2\theta\sin^4\theta) d\theta \\ &= \frac{3}{8}a^2\int_0^{2\pi} \sin^2 2\theta d\theta = \frac{3}{8}\pi a^2.\end{aligned}$$

Hence area is $3\pi a^2/8$. (Figure 15.16).

□

Vector Form using the Curl

Any vector field in $\mathbb{R}^2$ can be treated as a vector field in $\mathbb{R}^3$. For example, the vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ on $\mathbb{R}^2$ can be viewed as $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + 0\mathbf{k}$. Then we can define its curl and it can be shown that the curl is (compute!) $(\partial N/\partial x - \partial M/\partial y)\mathbf{k}$. Then we obtain

$$(\operatorname{curl}\mathbf{F})\cdot\mathbf{k} = \left[\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\right]\cdot\mathbf{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

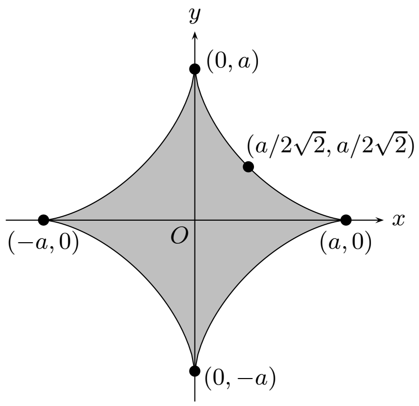


Figure 15.16: $x^{2/3} + y^{2/3} = a^{2/3}$

Hence by Green's theorem,

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{x} = \int_{\partial D} M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dxdy.$$

This is a vector form of Green's theorem.

Theorem 15.4.10. ***(Vector form of Green's theorem)*** *Let $D \subset \mathbb{R}^2$ be region with ∂D. If $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is a $\mathcal{C}^1$-vector field on D then*

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{x} = \iint_D (\mathbf{curl\ F}) \cdot \mathbf{k}\, dxdy = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dxdy.$$

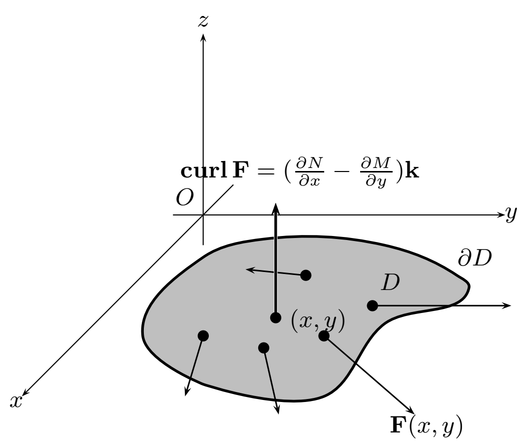


Figure 15.17: **curl F** is normal to the plane

Divergence Theorem - revisited

Definition 15.4.11. The **divergence density** of the vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is

$$\operatorname{div}\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

Theorem 15.4.12. ***(Green's theorem : Flux-Divergence form)*** *If* $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ *is a* $\mathcal{C}^1$*-vector field on* D *then*

$$\int_{\partial D} (\mathbf{F}\cdot\mathbf{n})ds = \int_{\partial D} M dy - N dx = \iint_D (\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y})\, dxdy = \iint_D \operatorname{div}\mathbf{F}\, dxdy.$$

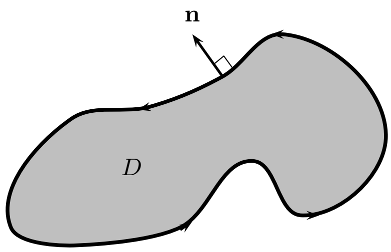


Figure 15.18: $\mathbf{n}$ is the unit outward normal vector to ∂D

Proof. Let $\mathbf{x}(t)$ be a parametrization of the boundary of D. Since $\mathbf{x}'(t) = (x'(t), y'(t))$ is tangent to ∂D we see $\mathbf{n}\cdot\mathbf{x}'(t) = 0$. i.e, $\mathbf{n}$ is perpendicular to the boundary. Choosing the proper sign of $\mathbf{n}$, we see

$$\mathbf{n} = \frac{(y'(t), -x'(t))}{\|\mathbf{x}'(t)\|}.$$

Hence

$$\begin{aligned}
\int_{\partial D} (\mathbf{F}\cdot\mathbf{n})ds &= \int_a^b \left(\frac{M(x,y)y'(t) - N(x,y)x'(t)}{\sqrt{[x'(t)]^2 + [y'(t)]^2}} \right) \sqrt{[x'(t)]^2 + [y'(t)]^2} dt \\
&= \int_a^b [M(x,y)y'(t) - N(x,y)x'(t)]dt \\
&= \int_{\partial D} M dy - N dx.
\end{aligned}$$

By Green's theorem,

$$\int_{\partial D}(\mathbf{F}\cdot\mathbf{n})ds = \int_{\partial D} M\,dy - N\,dx = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$
$$= \iint_D \operatorname{div}\mathbf{F}\,dxdy.$$

□

Summary of Green's theorem

2-D Green's theorem has several different forms:

$$\int_{\partial D} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy \tag{15.13}$$

$$\int_{\partial D}(\mathbf{F}\cdot\mathbf{T})ds = \int_{\partial D}\mathbf{F}\cdot d\mathbf{r} = \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dxdy \tag{15.14}$$

$$\int_{\partial D} M\,dy - N\,dx = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy \tag{15.15}$$

$$\int_{\partial D}(\mathbf{F}\cdot\mathbf{n})ds = \iint_D \operatorname{div}\mathbf{F}\,dxdy. \tag{15.16}$$

We will see the forms (15.14) and (15.16) will have a natural generalization to 3-D, called Stokes' theorem and Gauss divergence theorem respectively.

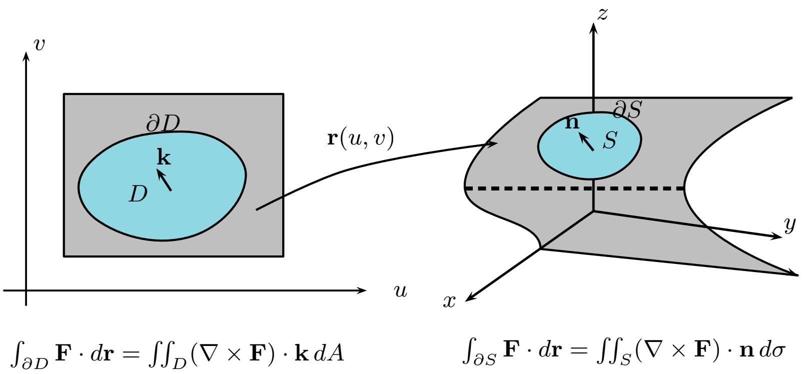


Figure 15.19: Green's theorem (15.14) to parameterized surface

15.5 (Parameterized) Surfaces and Surface area

Graphs are too restrictive.

Consider the surface of a sphere or a torus. These are important examples of figures that arise often in applications. But those figures cannot be represented as the graph of some functions. (refer to Fig. 15.19 or 15.20) Thus we need some other ways of representing surfaces.

Definition 15.5.1. A parameterized surface is a (one-to-one) function $\mathbf{r}\colon D \subset \mathbb{R}^2 \to \mathbb{R}^3$ where D is a domain in $\mathbb{R}^2$. The **underlying surface** S is the image $\mathbf{r}(D)$ of $\mathbf{r}$. The function $\mathbf{r}$ is also called a **parametrization** of S. Usually, we write

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

Remark 15.5.2. In the book they use the notation

$$x = f(u, v),\ \ y = g(u, v),\ \ z = h(u, v)$$

and set $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$.

If $\mathbf{r}$ is differentiable or C^1, then we say S is differentiable or C^1-surface.

Example 15.5.3. The graph of a function is a special case. If $z = f(x, y)$ $(x, y) \in D$ then

$$\mathbf{r}(u, v) = (u, v, f(u, v))$$

is a parametrization of the surface.

Example 15.5.4. Find a parametrization of the cone

$$z = \sqrt{x^2 + y^2},\ \ 0 \le z \le 1.$$

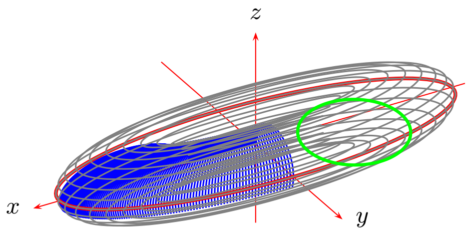
sol. The surface satisfies the equation

$$x = u \cos v,\ y = u \sin v,\ z = u,\ \ 0 \le u \le 1,\ \ 0 \le v \le 2\pi.$$

□

Example 15.5.5. (1) Let $D = [0, \pi] \times [0, 2\pi)$ and

$$\mathbf{r}(u, v) = (a \sin u \cos v, a \sin u \sin v, a \cos v).$$



$((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$

Figure 15.20: Surface of a torus

The parametric surface is a sphere of radius a.

(2) We set $\mathbf{r}(s, v) = (a\cos u, a\sin u, v)$, $0 \le u \le 2\pi$. This is a cylinder of radius a.

Example 15.5.6. Consider a parametrization of the surface.

$$\begin{cases} x = (a + b\cos\phi)\cos\theta, & 0 \le \theta, \phi \le 2\pi, \\ y = (a + b\cos\phi)\sin\theta, & a > b > 0, \\ z = b\sin\phi. \end{cases}$$

Investigate it.

$\boxed{\textbf{sol.}}$ Since $x^2 + y^2 = (a + b\cos\phi)^2$ it is easy to see this surface satisfies the equation

$$\left(\sqrt{x^2 + y^2} - a\right)^2 + z^2 = b^2.$$

Let us fix $\phi = \phi_0$. Then $z = b\sin\phi_0$ and hence it describes a circle (red color) of radius $a + b\cos\phi_0$ lying in the plane: $z = b\sin\phi_0$.

On the other hand, let us fix $\theta = \theta_0$. Then

$$\begin{cases} x = (a + b\cos\phi)\cos\theta_0, \\ y = (a + b\cos\phi)\sin\theta_0, \\ z = b\sin\phi \end{cases} \Rightarrow \begin{cases} x - a\cos\theta_0 = b\cos\phi\cos\theta_0, & (0 \le \phi \le 2\pi) \\ y - a\sin\theta_0 = b\cos\phi\sin\theta_0, \\ z = b\sin\phi. \end{cases}$$

Since

$$(x - a\cos\theta_0)^2 + (y - a\sin\theta_0)^2 + z^2 = b^2$$

the curve is a circle (green color) lying in the sphere of radius b, centered at $(a\cos\theta_0, -a\sin\theta_0, 0)$ determined by the plane $\theta = \theta_0$. (This is the plane $y = \tan\theta_0 x$.) This surface is called a **torus.**

$\square$

Normal Vectors, Tangent Planes, and Surface Area

Consider the mapping $\mathbf{r}\colon D \to \mathbb{R}^3$, where we write $\mathbf{r} = (x, y, z)$. First look at the case when the surface is the graph of $f\colon D \to \mathbb{R}$. Then we have

$$\mathbf{r}(x, y) = (x, y, f(x, y)).$$

To study the surface we look at the sections: First fix $y = y_0$ and then $x = x_0$.

Then the derivatives of $\mathbf{r}$ in the direction of x-axis and y-axis at $\mathbf{r}(x_0, y_0) = (x_0, y_0, f(x_0, y_0))$ are

$$\mathbf{r}_x(x_0, y_0) = \mathbf{i} + f_x(x_0, y_0)\mathbf{k}, \quad \mathbf{r}_y(x_0, y_0) = \mathbf{j} + f_y(x_0, y_0)\mathbf{k}.$$

These are nothing but the tangent vectors to the curves $\mathbf{r}(x, y_0)$ and $\mathbf{r}(x_0, y)$, respectively. Hence the normal vector is given by the cross product

$$\begin{aligned}
\mathbf{r}_x(x_0, y_0) \times \mathbf{r}_y(x_0, y_0) &= (\mathbf{i} + f_x(x_0, y_0)\mathbf{k}) \times (\mathbf{j} + f_y(x_0, y_0)\mathbf{k}) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x_0, y_0) \\ 0 & 1 & f_y(x_0, y_0) \end{vmatrix} \\
&= -f_x(x_0, y_0)\mathbf{i} - f_y(x_0, y_0)\mathbf{j} + \mathbf{k}.
\end{aligned}$$

In general, consider the surface parameterized by

$$\mathbf{r}(x(u, v), y(u, v)) = (x(u, v), y(u, v), z(u, v)).$$

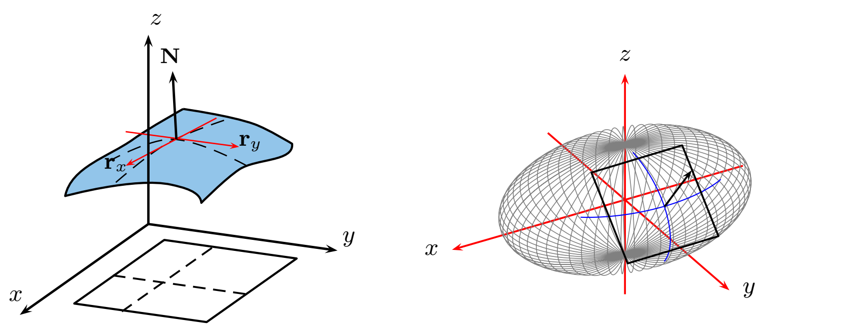
Then we see two tangent vectors are

$$\begin{aligned}
\mathbf{r}_u &= \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\Big|_{(u_0, v_0)} \\
\mathbf{r}_v &= \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\Big|_{(u_0, v_0)}
\end{aligned}$$

These are obtained by considering the cross sections with the planes $v = v_0$ and $u = u_0$, respectively. If the normal vector

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$$

is nonzero, then we say the surface is **smooth**.



ellipsoid: $(a \sin\phi \cos\theta, b \sin\phi \sin\theta, c\cos\phi)$

Figure 15.21: Coord. curves, tangent vectors and normal vectors to a surface

Definition 15.5.7. When $\mathbf{N}$ is a normal vector to a surface $\mathbf{r}$, the **tangent plane** at $\mathbf{r}(u_0, v_0) = (x_0, y_0, z_0)$ is defined by

$$\mathbf{N} \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

If $\mathbf{N} = (n_1, n_2, n_3)$, then the equation of **tangent plane** is

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0.$$

Example 15.5.8. Consider the surface given by

$$x = u\cos v, \quad y = u \sin v, \quad z = u^2 + v^2.$$

Find the tangent plane at $\mathbf{r}(1, 0)$.

$\boxed{\text{sol.}}$ Since $\mathbf{r}(u, v) = (u\cos v, u \sin v, u^2 + v^2)$ we have

$$\mathbf{r}_v = (\cos v, \sin v, 2u), \quad \mathbf{r}_v = (-u \sin v, u\cos v, 2v).$$

Hence we see $\mathbf{r}_u \times \mathbf{r}_v = (-2u^2 \cos v + 2v \sin v, -2u^2 \sin v - 2v \cos v, u)$. Since $\mathbf{r}(1,0) = (1,0,1)$ and $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v|(1,0) = (-2,0,1)$, we see the tangent plane is given as

$$-2(x-1) + 0(y-0) + 1(z-1) = 0.$$

□

Example 15.5.9. Find a parametrization of the following hyperboloid of one sheet

$$x^2 + y^2 - z^2 = 1.$$

sol. Since the graph is symmetric in x and y, it is natural to use polar coordinate

$$x = r\cos\theta, \quad y = r\sin\theta, \quad (0 \le \theta < 2\pi)$$

to transform it to

$$r^2 - z^2 = 1.$$

Next we use the following parametrization

$$r = \cosh s, \quad z = \sinh s, \quad (-\infty < s < \infty)$$

to get

$$x = \cosh s \cos\theta, \quad y = \cosh s \sin\theta, \quad z = \sinh s.$$

So

$$\begin{aligned}\mathbf{r}(s,\theta) &= (x(s,\theta), y(s,\theta), z(s,\theta))\\ &= (\cosh s\cos\theta, \cosh s\sin\theta, \sinh s),\ (-\infty < s < \infty, 0 \le \theta < 2\pi).\end{aligned}$$

□

Area of Parameterized Surface

Recall 2-D case: When $\mathbf{r} : D \to R$ is a transformation in $\mathbb{R}^2$. Consider the small rectangle $A = [u, u + \Delta u] \times [v + \Delta v]$. The two tangent vectors $(\Delta u, 0)$ and $(0, \Delta v)$ are mapped to the boundary of image $\mathbf{r}(A)$ at $\mathbf{r}(u,v)$ as

$$\mathbf{r}_u \Delta u, \quad \mathbf{r}_v \Delta v.$$

These vectors form a parallelogram approximating the region $\mathbf{r}(A)$(figure 15.22). The area of the parallelogram is

$$\begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v = \frac{\partial(x,y)}{\partial(u,v)}\Delta u \Delta v.$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| \Delta u \Delta v = |J| \Delta u \Delta v.$$

Hence we have

$$\iint_R dxdy = \iint_D |J| dudv.$$

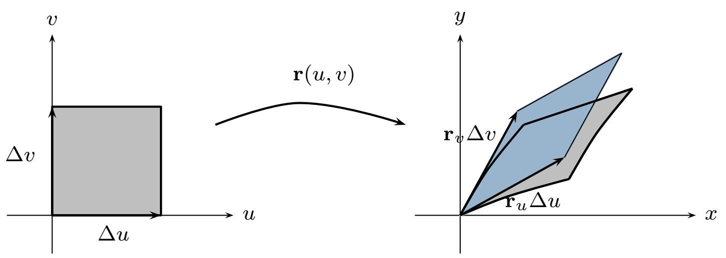


Figure 15.22: approximate $\mathbf{r}(A)$

Now we consider a surface lying in space. We will show how to find the area of $S = \mathbf{r}(D)$ where $\mathbf{r}\colon D \to \mathbb{R}^3$ is a surface parametrization. First divide the domain D into small rectangles. Consider a small rectangle $A = [u, u + \Delta u] \times [v, v + \Delta v]$. The image of A under $\mathbf{r}$ is a portion of the surface having four corners at

$$\mathbf{r}(u, v), \quad \mathbf{r}(u + \Delta u, v), \quad \mathbf{r}(u, v + \Delta v), \quad \mathbf{r}(u + \Delta u, v + \Delta v).$$

This surface can be approximated by a parallelogram whose sides are given by(fig 15.23) $\mathbf{r}_u(u, v)\Delta u$ and $\mathbf{r}_v(u, v)\Delta v$, where

$$\begin{aligned} \mathbf{r}_u &= \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \\ \mathbf{r}_v &= \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}. \end{aligned} \tag{15.17}$$

Hence the area of $\mathbf{r}(A)$ is (again like 2D) approximated by

$$\|\mathbf{r}_u \times \mathbf{r}_v\| \Delta u \Delta v.$$

Hence the area of the surface is the limit of the following sum:

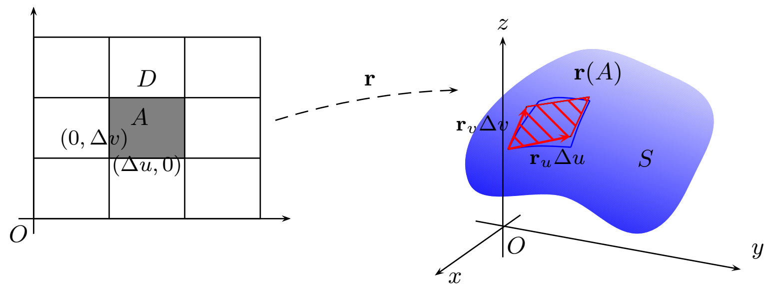


Figure 15.23: Approx. area of surface by a tangent plane

$$\sum \|\mathbf{r}_u \times \mathbf{r}_v\| \Delta u \Delta v.$$

Definition 15.5.10. We define the surface area $A(S)$ of a parameterized surface $S = \mathbf{r}(D)$ by

$$A(S) = \iint_S dS = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| dudv.$$

We let

$$d\sigma = dS = \|\mathbf{r}_u \times \mathbf{r}_v\| dudv,$$

and call it the **surface area differential**. Then we see that [2]

$$\iint_S d\sigma = \iint_S dS = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| dudv.$$

Remark 15.5.11. The area of a surface is independent of parametrization.

Example 15.5.12 (Cone)**.** Let S be the surface of a cone given by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r, \ 0 \le r \le 1.$$

$\boxed{\textbf{sol.}}$ Either use formula above or compute directly using $\|\mathbf{r}_r \times \mathbf{r}_\theta\| dr d\theta$. We

[2] $\mathbf{r}$ is assumed to be 1-1.

can show that $\|\mathbf{r}_r \times \mathbf{r}_\theta\| = r\sqrt{2}$. Hence the area is

$$\begin{aligned}\iint_S d\sigma &= \iint_D \|\mathbf{r}_r \times \mathbf{r}_\theta\| dr d\theta \\ &= \iint_D r\sqrt{2} dr d\theta \\ &= \int_0^{2\pi} \int_0^1 r\sqrt{2} dr d\theta \\ &= \int_0^{2\pi} \frac{\sqrt{2}}{2} d\theta = \pi\sqrt{2}.\end{aligned}$$

□

Example 15.5.13 (Football like surface)**.** Find the area of the surface of revolution of the curve $x = \cos z, y = 0, |z| \le \pi/2$ around z-axis.

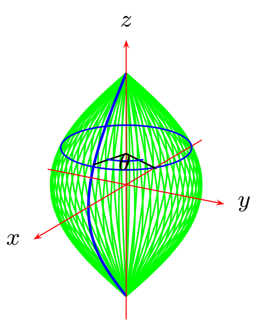
sol. The surface of revolution is parameterized by

$$\mathbf{r}(u,v) = (x,y,z),\ \ x = \cos u \cos v,\ \ y = \cos u \sin v,\ \ z = u, |u| \le \frac{\pi}{2},\ \ 0 \le v \le 2\pi.$$

We see

$$\begin{aligned}\mathbf{r}_u &= -\sin u \cos v\mathbf{i} - \sin u \sin v\mathbf{j} + \mathbf{k} \\ \mathbf{r}_v &= -\cos u \sin v\mathbf{i} + \cos u \cos v\mathbf{j}.\end{aligned}$$

Compute $\|\mathbf{r}_r \times \mathbf{r}_\theta\|$.



$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u \cos v & -\sin u \sin v & 1 \\ -\cos u \sin v & \cos u \cos v & 0 \end{vmatrix} \\ &= -\cos u \cos v\mathbf{i} - \cos u \sin v\mathbf{j} - \sin u \cos u\mathbf{k} \\ \|\mathbf{r}_u \times \mathbf{r}_v\| &= \cos u\sqrt{1 + \sin^2 u}\end{aligned}$$

Hence the area is

$$\begin{aligned} A &= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \cos u\sqrt{1+\sin^2 u}\, dudv \\ &= 2\int_0^{2\pi}\int_0^{\pi/2} \sqrt{1+t^2}\, dtdv(\text{ need table}) \\ &= \int_0^{2\pi} \left[t\sqrt{1+t^2} + \ln(t+\sqrt{1+t^2})\right]_0^1 dv \\ &= 2\pi\left[\sqrt{2}+\ln(1+\sqrt{2})\right]. \end{aligned}$$

□

Example 15.5.14 (Helicoid-like surface)**.** Let S be the surface given by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = \theta, (0 \le \theta \le 2\pi,\ 0 \le r \le 1).$$

Find its area.

sol. $\|\mathbf{r}_r \times \mathbf{r}_\theta\| = \|(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \times (-r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j})\| = \sqrt{r^2+1}.$

$$\begin{aligned} A &= \int_0^{2\pi}\int_0^1 \sqrt{1+r^2} drd\theta(\text{ as in the previous example}) \\ &= \frac{1}{2}\int_0^{2\pi} \left[r\sqrt{1+r^2} + \ln(r+\sqrt{1+r^2})\right]_0^1 d\theta \\ &= \pi\left[\sqrt{2}+\ln(1+\sqrt{2})\right]. \end{aligned}$$

□

Implicit Surfaces

Assume a surface is defined implicitly by

$$F(x, y, z) = c. \tag{15.18}$$

In this case, it is not easy to find the explicit form of parametrization. However, we can still compute

$$d\sigma = dS = \left\|(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}) \times (\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k})\right\| dudv \tag{15.19}$$

from the implicit expression. Assume the surface is defined over a region R having $\mathbf{k}$ as the unit normal vector. Define the parameters $x = u, y = v$ then

$z(x, y) = z(u, v)$. It has the surface has the following parametrization

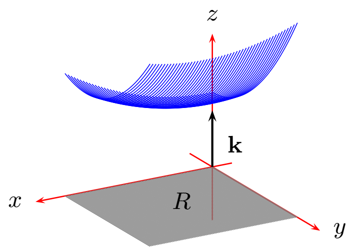


Figure 15.24: Implicit surface $F(x, y, z) = c$ with normal vector $\mathbf{k}$ on R

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z(x, y)\mathbf{k}. \tag{15.20}$$

Then

$$\mathbf{r}_x = \mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k} \text{ and } \mathbf{r}_y = \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k}. \tag{15.21}$$

Meanwhile, taking the implicit derivative of (15.18) w.r.t x and y, we get

$$F_x + F_z\frac{\partial z}{\partial x} = 0 \text{ and } F_y + F_z\frac{\partial z}{\partial y} = 0.$$

From this we get

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Substituting these into (15.21), we get

$$\mathbf{r}_x = \mathbf{i} - \frac{F_x}{F_z}\mathbf{k} \text{ and } \mathbf{r}_y = \mathbf{j} - \frac{F_y}{F_z}\mathbf{k} \tag{15.22}$$

and

$$\begin{aligned}
\mathbf{r}_x \times \mathbf{r}_y &= \frac{F_x}{F_z}\mathbf{i} + \frac{F_y}{F_z}\mathbf{j} + \mathbf{k} \\
&= \frac{1}{F_z}(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \\
&= \frac{\nabla F}{F_z} = \frac{\nabla F}{\nabla F \cdot \mathbf{k}} \\
\therefore \quad d\sigma &= \|\mathbf{r}_u \times \mathbf{r}_v\| du dv.
\end{aligned}$$

When R is a region having $\mathbf{i}$ or $\mathbf{j}$ as the normal vector, the same argument applies. Thus we have

The area of implicit surface $F(x, y, z) = c$ defined over R is

$$\iint_R d\sigma = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA,$$

where $\mathbf{p} = \mathbf{i}, \mathbf{j}$ or $\mathbf{k}$ is the normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

Example 15.5.15. Find the area of surface of paraboloid $x^2 + y^2 - z = 0$ between $0 \le z \le 4$.

sol. Let $F(x, y, z) = x^2 + y^2 - z$ so that $\nabla F = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$. $\nabla F \cdot \mathbf{k} = -1$. With $D = \{x^2 + y^2 \le 4\}$, the area is

$$\begin{aligned}
A &= \iint_D \sqrt{4x^2 + 4y^2 + 1} dxdy \\
&= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1} r dr d\theta \\
&= \int_0^{2\pi} \frac{1}{12} \left[(4r^2 + 1)^{3/2} \right]_0^2 d\theta \\
&= \frac{\pi}{6}(17\sqrt{17} - 1).
\end{aligned}$$

□

Surface Area of a Graph

When a surface S is given by the graph of function $z = f(x, y)$ on D, we see U is parameterized by $\mathbf{r}(x, y) = (x, y, f(x, y))$. Find $\mathbf{r}_x$, $\mathbf{r}_y$ by

$$\mathbf{r}_x = \mathbf{i} + f_x\mathbf{k}, \qquad \mathbf{r}_y = \mathbf{j} + f_y\mathbf{k}.$$

This corresponds to above case with $F(x, y, z) = z - f(x, y)$.

Since

$$\mathbf{r}_x \times \mathbf{r}_y = (\mathbf{i} + f_x\mathbf{k}) \times (\mathbf{j} + f_y\mathbf{k}) = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k},$$

the area is

$$\iint_S d\sigma = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1} dxdy.$$

Geometric interpretation

We refer to figure 15.25. The unit normal vector $\mathbf{N}(x, y, z)$ on S is

$$\mathbf{N}(x, y, z) = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}.$$

We can find the formula using the angle between $\mathbf{N}$ and $\mathbf{k}$. Let θ be the angle between $\mathbf{N}$ and $\mathbf{k}$. Then $\cos\theta$ satisfies

$$\cos\theta = \frac{\mathbf{N}\cdot\mathbf{k}}{\|\mathbf{N}\|} = \frac{1}{\sqrt{(f_x)^2 + (f_y)^2 + 1}}.$$

Hence

$$d\sigma = \sqrt{(f_x)^2 + (f_y)^2 + 1}dxdy = \frac{dxdy}{\cos\theta},$$

and we get

$$\iint_S d\sigma = \iint_D \frac{dxdy}{\cos\theta}.$$

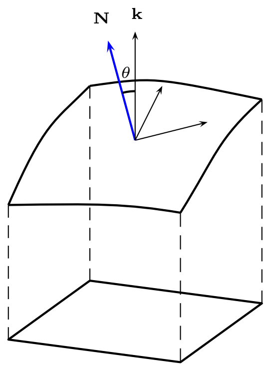


Figure 15.25: Ratio between two surface area is the cosine of angle

Example 15.5.16. Find the surface area of a unit ball.

sol. From $x^2 + y^2 + z^2 = 1$, we let $z = f(x, y) = \sqrt{1 - x^2 - y^2}$.

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{1 - x^2 - y^2}}, \quad \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1 - x^2 - y^2}}.$$

Area of the half sphere is

$$\begin{aligned}\iint_S d\sigma &= \iint_D \frac{1}{\sqrt{1-x^2-y^2}}dxdy\\ &= \int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{1-r^2}}\,drd\theta\\ &= 2\pi.\end{aligned}$$

□

Example 15.5.17. Let $\mathbf{r} = (r\cos\theta, r\sin\theta, \theta)$ be the parametrization of a helicoid-like surface S, where $0 \le r \le 1$, $0 \le \theta \le 2\pi$. Suppose S is covered with a metal of density m which equal to twice the distance to the central axis, i.e, $m = 2\sqrt{x^2+y^2} = 2r$. Find the total mass of metal covering the surface.

sol. First we can show $\|\mathbf{r}_r \times \mathbf{r}_\theta\| = \sqrt{1+r^2}$. Hence we have

$$\begin{aligned}M &= \iint_S 2rd\sigma = 2\iint_D r\|\mathbf{r}_r \times \mathbf{r}_\theta\|drd\theta\\ &= \int_0^{2\pi}\int_0^1 2r\sqrt{1+r^2}drd\theta = \frac{4}{3}\pi(2^{3/2}-1).\end{aligned}$$

□

15.6 Surface Integrals

Integrals of scalar functions over Surface

Let $\mathbf{r}\colon D \to \mathbb{R}^3$ be a parameterized surface $S = \mathbf{r}(D)$ and let $f\colon S \to \mathbb{R}$ be a real valued function defined on $\mathbf{r}$. If $f = 1$, it represents the area:

$$\iint_S d\sigma = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|dudv.$$

In general, we have

Definition 15.6.1. Let S be a surface parameterized by $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$, where $(u,v) \in D$. Then the surface integral of a scalar function $f(x,y,z)$ defined on S is

$$\iint_S f\,d\sigma = \iint_D f(x(u,v), y(u,v), z(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|dudv.$$

Surface integrals over graphs

Suppose S is the graph of a C^1 function $z = g(x, y)$. Then we parameterize it by

$$x = u, \quad y = v, \quad z = g(u, v)$$

and

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{1 + (g_u)^2 + (g_v)^2}.$$

So the integral of f on S becomes

$$\iint_S f(x, y, z)\, d\sigma = \iint_D f(x, y, g(x, y))\sqrt{1 + (g_x)^2 + (g_y)^2}\, dxdy.$$

Example 15.6.2. Let S be graph of $z = x^2 + y$, where D is $0 \le x \le 1, \ -1 \le y \le 1$. Find $\iint_S x\, dS$.

sol.

$$\begin{aligned}
\iint_S x\, d\sigma &= \iint_D x\sqrt{1 + (g_x)^2 + (g_y)^2}\, dxdy = \int_{-1}^{1}\int_0^1 x\sqrt{1 + 4x^2 + 1}dxdy \\
&= \frac{1}{8}\int_{-1}^{1}\left[\int_0^1 (2 + 4x^2)^{1/2}(8xdx)\right] dy = \frac{2}{3}\frac{1}{8}\int_{-1}^{1}\left[(2 + 4x^2)^{3/2}\right]\Big|_0^1 dy \\
&= \sqrt{6} - \frac{\sqrt{2}}{3}.
\end{aligned}$$

□

Example 15.6.3. Evaluate $\iint_S z^2 dS$ when S is the unit sphere.

sol. The unit sphere is described by

$$\mathbf{r}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \ (0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi).$$

Since

$$\|\mathbf{r}_\phi \times \mathbf{r}_\theta\| = \sin\phi$$

and $z^2 = \cos^2\phi$, we have

$$\begin{aligned}\iint_S z^2 d\sigma &= \iint_D \cos^2\phi \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} \cos^2\phi \sin\phi \, d\phi d\theta \\ &= \frac{4\pi}{3}.\end{aligned}$$

□

Example 15.6.4. Evaluate $\iint_S G(x,y,z)d\sigma$ over a football like surface S

$$x = \cos u \cos v,\ \ y = \cos u \sin v,\ \ z = u, -\frac{\pi}{2} \le u \le \frac{\pi}{2}, 0 \le v \le 2\pi$$

when $G(x,y,z) = \sqrt{1-x^2-y^2}$.

sol. Over the football surface the function G is given by

$$\sqrt{1-x^2-y^2} = \sqrt{1-\cos^2 u} = |\sin u|.$$

The surface differential is (Ref. Example 15.5.13)

$$d\sigma = \cos u\sqrt{1+\sin^2 u}\, dudv.$$

Hence

$$\begin{aligned}\iint_S \sqrt{1-x^2-y^2}d\sigma &= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} |\sin u| \cos u\sqrt{1+\sin^2 u} dudv \\ &= 2\int_0^{2\pi}\int_0^{\pi/2} \sin u\cos u\sqrt{1+\sin^2 u} dudv (w = 1+\sin^2 u) \\ &= \int_0^{2\pi}\int_1^2 \sqrt{w}dwdv \\ &= 2\pi\cdot\frac{2}{3}w^{3/2}|_1^2 = \frac{4\pi}{3}(2\sqrt{2}-1).\end{aligned}$$

□

Example 15.6.5. Evaluate $\iint_S \sqrt{x(1+2z)}dS$ where $S = \{z = y^2/2, x, y \ge 0,\ x+y \le 1\}$.

sol. This is an integral over a graph of a function. Let $z = g(x, y) = y^2/2$ so that the surface differential is

$$d\sigma = \sqrt{g_x^2 + g_y^2 + 1}dxdy = \sqrt{y^2 + 1}dxdy.$$

The surface integral is

$$\begin{aligned}\iint_S \sqrt{x(1+2z)}\sqrt{y^2+1}dxdy &= \int_0^1 \int_0^{1-x} \sqrt{x}(y^2+1)dydx \\ &= \int_0^1 \sqrt{x}((1-x) + \frac{1}{3}(1-x)^3)dx = \frac{284}{945}.\end{aligned}$$

□

Orientation

As in the case of line integral, the surface integral also has the notion of direction. First we need to define the **orientation** of a surface S. It depends on the particular parametrization.

Definition 15.6.6 (Oriented Surface)**.** An **orientable surface** is a two sided surface with one side specified as outside(or positive side). For orientable surface, there are two possible normal vectors at each point, i.e, two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, where $\mathbf{n}_1 = -\mathbf{n}_2$. Each of these normal vector can be associated with an **orientation**. There are nonorientable surfaces.(Example: Möbius strip)

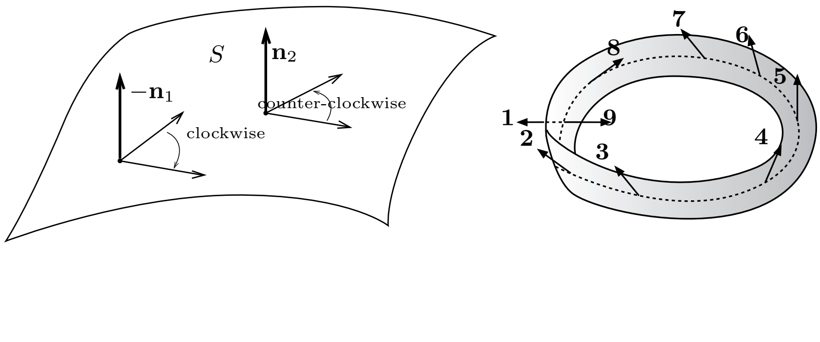
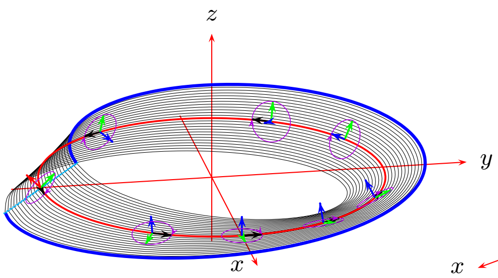
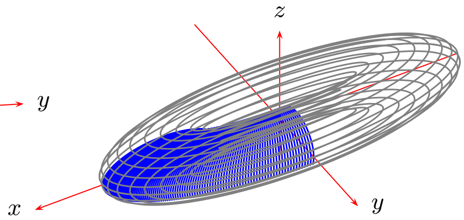


Figure 15.26: Orientable surface with two sides and Möbius strip



$((1+\frac{u}{2}\cos\frac{t}{2})\cos t,\ (1+\frac{u}{2}\cos\frac{t}{2})\sin t,\ \frac{u}{2}\sin\frac{t}{2})$



$((a+b\cos\phi)\cos\theta, (a+b\cos\phi)\sin\theta, b\sin\phi)$

Figure 15.27: Graph of Möbius strip and torus

Let $\mathbf{r}\colon D \to \mathbb{R}^3$ represent an oriented surface. If $\mathbf{n}(\mathbf{r})$ is the unit normal to S, then

$$\mathbf{n}(\mathbf{r}) = \pm\frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}.$$

We choose a parametrization so that the sign is positive (**orientation-preserving**)

Example 15.6.7. The parametrization of sphere by spherical coordinate by (ρ, ϕ, θ) is orientation-preserving. By changing the order of θ and ϕ, we can get orientation-reversing parametrization.

Example 15.6.8 (Möbius strip: non-orientable surface)**.** Consider the surface given by the following :

$$\begin{cases} x = \left(1 + v\cos\frac{u}{2}\right)\cos u \\ y = \left(1 + v\cos\frac{u}{2}\right)\sin u, \ \ 0 \le u \le 2\pi, \ \ -\frac{1}{2} \le v \le \frac{1}{2} \\ z = v\sin\frac{u}{2} \end{cases}.$$

Let $u = u_0$. Then

$$\begin{cases} x = \left(\cos u_0 \cos\frac{u_0}{2}\right) v + \cos u_0 \\ y = \left(\sin u_0 \cos\frac{u_0}{2}\right) v + \sin u_0, \ \ -\frac{1}{2} \le v \le \frac{1}{2} \\ z = \left(\sin\frac{u_0}{2}\right) v \end{cases}.$$

Orientation of a graph

Example 15.6.9. Let S be the graph of a function $z = g(x, y)$. Usually, we give the orientation of such surface by taking the **positive side** to be the side away from which $\mathbf{n}$ points. Then the unit normal is given by

$$\mathbf{n} = \frac{-g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + (g_x)^2 + (g_y)^2}}\, dxdy.$$

Surface Integrals of vector Fields

In this section we develop the notion of integral of a vector field over a surface.

Recall the line integral of a vector field has a physical interpretation: **Work**. Similarly, the notion of integral of a vector field over a surface can be interpreted as a **Flux**.

Assume the vector field $\mathbf{F}\colon V \to \mathbb{R}^3$ represents the velocity of a fluid and the parametrization $\mathbf{r}\colon D \to V \subset \mathbb{R}^3$ describes the shape of the net. Then the surface integral $\iint_{\mathbf{r}(D)} \mathbf{F}\cdot\mathbf{n}d\sigma$ is the **amount of fluid that passes through the surface (per unit time)**.

We now define the surface integral of $\mathbf{F}$ over a surface S:

Definition 15.6.10.

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS.$$

In other word, the surface integral of $\mathbf{F}$ on a surface S is the surface integral of normal projection of $\mathbf{F}$ to the surface S.

Since $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v/\|\mathbf{r}_u \times \mathbf{r}_v\|$ is the unit normal vector to the surface,

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma &= \iint_D \mathbf{F}\cdot\mathbf{n}\|\mathbf{r}_u \times \mathbf{r}_v\| dudv \\
&= \iint_D \mathbf{F}\cdot\frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}\|\mathbf{r}_u \times \mathbf{r}_v\|\, dudv \\
&= \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dudv \\
&:= \iint_{S=\mathbf{r}(D)} \mathbf{F}\cdot d\boldsymbol{\sigma}.
\end{aligned}$$

Here

$$\boxed{d\boldsymbol{\sigma} = \mathbf{n}\, d\sigma = (\mathbf{r}_u \times \mathbf{r}_v)\, dudv} \qquad (15.23)$$

is the vector version of surface element $d\sigma$ given in the Definition 15.6.1, but different in that a normal vector is attached.

Example 15.6.11. Find the flux of $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$ through the surface

S given by

$$y = x^2,\ 0 \le x \le 1,\ 0 \le z \le 4.$$

sol. We can parameterize the surface using (x, z). $\mathbf{r} = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$. So

$$\begin{aligned}\mathbf{r}_x &= \mathbf{i} - 2x\mathbf{j}, \quad \mathbf{r}_z = \mathbf{k} \\ \mathbf{r}_x \times \mathbf{r}_z &= 2x\mathbf{i} - \mathbf{j} \\ \mathbf{n} &= \frac{2x\mathbf{i} - \mathbf{j}}{\sqrt{4x^2+1}}.\end{aligned}$$

On the surface

$$\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k} = x^2z\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}.$$

Hence

$$\begin{aligned}\mathbf{F} \cdot \mathbf{n} &= \frac{1}{\sqrt{4x^2+1}}(x^2z \cdot 2x - x) \\ &= \frac{2x^3z - x}{\sqrt{4x^2+1}}, \\ \iint_{\mathbf{r}(D)} \mathbf{F} \cdot \mathbf{n} d\sigma &= \int_0^4 \int_0^1 \frac{2x^3z - x}{\sqrt{4x^2+1}} \|\mathbf{r}_x \times \mathbf{r}_z\| dxdz \\ &= \int_0^4 \int_0^1 (2x^3z - x)xdz \\ &= 2.\end{aligned}$$

or directly, we can integrate

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_D (x^2z\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}) \cdot (2x\mathbf{i} - \mathbf{j}) dxdz = \iint_D 2x^3z - xdxdz.$$

□

Example 15.6.12 (Spherical coordinate)**.** Let S be the unit sphere parameterized by

$$\mathbf{r}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),\ (0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi).$$

Compute $\iint_S \mathbf{r} \cdot d\boldsymbol{\sigma}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}$ denotes the position vector.

sol. We see

$$\begin{aligned}\mathbf{r}_\phi &= \cos\phi\cos\theta\mathbf{i} + \cos\phi\sin\theta\mathbf{j} - \sin\phi\mathbf{k},\\ \mathbf{r}_\theta &= -\sin\phi\sin\theta\mathbf{i} + \sin\phi\cos\theta\mathbf{j},\\ \mathbf{r}_\phi \times \mathbf{r}_\theta &= \sin\phi(\cos\theta\sin\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\phi\mathbf{k}).\end{aligned}$$

Hence $\mathbf{r}\cdot d\boldsymbol{\sigma} = \mathbf{r}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta)d\phi\, d\theta = \sin\phi\, d\phi\, d\theta$ and

$$\iint_{\mathbf{r}(D)} \mathbf{r}\cdot d\boldsymbol{\sigma} = \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = 4\pi.$$

$\square$

Surface integrals of vector fields over an implicit surface $G(x,y,z) = 0$

The unit normal vector to the surface $S : G(x,y,z) = 0$ is

$$\mathbf{n} = \frac{\nabla G}{|\nabla G|}.$$

Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}d\boldsymbol{\sigma} = \iint_R \mathbf{F}\cdot\frac{\nabla G}{|\nabla G|}\frac{|\nabla G|}{|\nabla G\cdot\mathbf{p}|}dA = \iint_R \mathbf{F}\cdot\frac{\nabla G}{|\nabla G\cdot\mathbf{p}|}dudv,$$

where $\mathbf{p} = \mathbf{i},\mathbf{j}$ or $\mathbf{k}$ is the normal to R and $\nabla F\cdot\mathbf{p} \neq 0$.

Surface Integral of vector fields over Graphs

Suppose S is the graph of $z = g(x,y)$. We parameterize the surface S by $\mathbf{r}(x,y) = (x,y,g(x,y))$ and compute

$$\mathbf{r}_x = \mathbf{i} + g_x\mathbf{k}, \quad \mathbf{r}_y = \mathbf{j} + g_y\mathbf{k}.$$

Hence

$$\mathbf{r}_x \times \mathbf{r}_y = -(g_x)\mathbf{i} - (g_y)\mathbf{j} + \mathbf{k}$$

and we see

$$\iint_S \mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_D \mathbf{F}\cdot(\mathbf{r}_x \times \mathbf{r}_y)dxdy = \iint_D [F_1(-g_x) + F_1(-g_y) + F_3]\, dxdy.$$

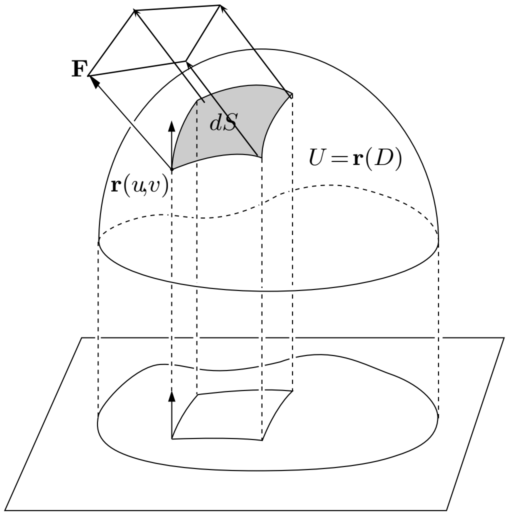


Figure 15.28: Area of shadow region and flux across S

Physical Interpretation of Surface Integrals

Consider the parallelepiped determined by three vectors $\mathbf{F}$, $\mathbf{r}_u\Delta u$ and $\mathbf{r}_v\Delta v$. (See figure 15.28.) Its volume is

$$\mathbf{F} \cdot (\mathbf{r}_u\Delta u \times \mathbf{r}_v\Delta v) = \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\Delta u\Delta v.$$

If $\mathbf{F}$ is the velocity of a fluid, the volume is *the amount of fluid flowing out of the surface per unit time.* Hence

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S \mathbf{F} \cdot d\boldsymbol{\sigma}$$

is the *net quantity of fluid to flow across the surface per unit time*, i.e, **the rate of fluid flow**. It is also called **flux of F across** S.

Example 15.6.13 (Heat flow)**.** Let T denote the temperature at a point. Then

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}$$

represents the temperature gradient and heat "flows" with the vector field

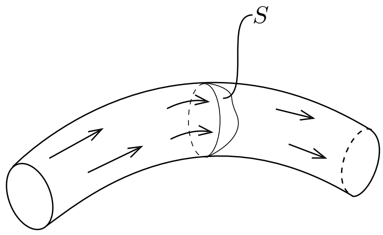


Figure 15.29: Water through a pipe and a surface S

$-k\nabla T$.

Example 15.6.14. Suppose temperature on a sphere $S\colon x^2+y^2+z^2=1$ is $T=x^2+y^2+z^2$. Find the total heat flux across S if $k=1$.

sol. We have heat flow $\mathbf{F}=-\nabla T=-2(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=-2\mathbf{r}$ and the unit normal vector to S is $\mathbf{n}=(x,y,z)=\mathbf{r}$. Hence

$$\iint_S \mathbf{F}\cdot d\boldsymbol{\sigma}=\iint_S(-2\mathbf{r}\cdot\mathbf{n})d\sigma=-2\iint_S dS=-8\pi.$$

□

Example 15.6.15 (Gauss Law)**.** The flux of an electric field $\mathbf{E}$ over a closed surface S is the net charge Q contained in the surface. Namely,

$$\iint_S \mathbf{E}\cdot d\boldsymbol{\sigma}=Q.$$

Suppose $\mathbf{E}=E\mathbf{n}$(constant multiple of the unit normal vector) then

$$\iint_S \mathbf{E}\cdot d\boldsymbol{\sigma}=\iint_S EdS=Q=E\cdot A(S).$$

So $E=\frac{Q}{A(S)}$ and if S is sphere of radius R then

$$E=\frac{Q}{4\pi R^2}. \tag{15.24}$$

Example 15.6.16. Given a disk lying on the plane $z=12$ described by

$$z=12,\quad x^2+y^2\le 25,$$

compute $\iint_S \mathbf{r}\cdot d\boldsymbol{\sigma}$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

sol. We see

$$\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} \times \mathbf{j} = \mathbf{k}.$$

So $\mathbf{r}\cdot(\mathbf{r}_x \times \mathbf{r}_y) = z$ and

$$\iint_S \mathbf{r}\cdot d\boldsymbol{\sigma} = \iint_D z dx dy = 12A(D) = 300\pi.$$

□

Mass and Moment for very thin shells

Imagine a very thin object like, drums, dome of a stadium. These materials can be treat like surfaces (having no thickness). Mass and moment for these thin shells can e computed as before.

$$\text{The total mass } M \quad = \quad \iint_S \delta dS.$$

The **moment about coord. plane**

$$M_{yz} = \iint_S x\delta\, d\sigma, \quad M_{zx} = \iint_S y\delta\, d\sigma, \quad M_{xy} = \iint_S z\delta\, d\sigma.$$

The **moment of inertia about coord. axis**

$$I_x = \iint_S (y^2 + z^2)\delta\, d\sigma, \quad I_y = \iint_S (x^2 + z^2)\delta\, d\sigma, \quad I_z = \iint_S (x^2 + y^2)\delta\, d\sigma$$

Example 15.6.17. Find the center of mass of a thin hemisphere shell of radius a and density δ.

sol. We see shell is described by

$$f(x, y, z) = x^2 + y^2 + z^2 = a^2, \ z \geq 0.$$

The symmetry tells us $\bar{x} = \bar{y} = 0$. To compute $\bar{z}$, we need M_{xy}/M. The mass is

$$\iint_S \delta\, d\sigma = \delta A(S) = 2\pi a^2 \delta.$$

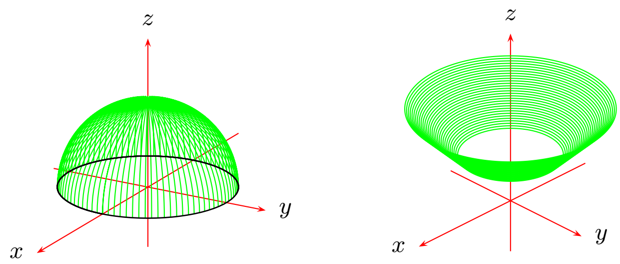


Figure 15.30: Hemisphere and part of a cone

$$\begin{aligned}
|\nabla f| &= |2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}| = 2\sqrt{x^2+y^2+z^2} = 2a \\
|\nabla f \cdot \mathbf{k}| &= |2z| = 2z \\
d\sigma &= \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} = \frac{a}{z} dA.
\end{aligned}$$

$$\begin{aligned}
M_{xy} &= \iint_S z\delta d\sigma = \delta \iint_S z\frac{a}{z} dA = \delta\pi a^3 \\
\bar{z} &= \frac{M_{xy}}{M} = \frac{a}{2}.
\end{aligned}$$

□

Example 15.6.18. Find the center of mass of a thin shell of density $\delta = 1/z^2$ cut from the cone $z = \sqrt{x^2+y^2}$ by the plane $z = 1$ and $z = 2$.

sol. The symmetry tells us $\bar{x} = \bar{y} = 0$.

We see the cone is described by parametrization $x = r\cos\theta,\ y = r\sin\theta, z = r$, i.e.,

$$\mathbf{r}(r,\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + r\mathbf{k}, \quad 1 \le r \le 2,\ 0 \le \theta \le 2\pi$$

and $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{2}r$.(early)

To compute $\bar{z}$, we need M_{xy}/M. The mass is

$$\begin{aligned}
M &= \iint_S \delta\, d\sigma = \int_0^{2\pi}\int_1^2 \frac{1}{r^2}\sqrt{2}r dr d\theta \\
&= \sqrt{2}\int_0^{2\pi} [\ln r]_1^2 d\theta \\
&= 2\pi\sqrt{2}\ln 2.
\end{aligned}$$

$$\begin{aligned} M_{xy} &= \iint_S z\delta d\sigma = \int_0^{2\pi}\int_1^2 \frac{1}{r^2} r\sqrt{2} r dr d\theta \\ &= \sqrt{2}\int_0^{2\pi}\int_1^2 dr d\theta = 2\pi\sqrt{2} \\ \bar{z} &= \frac{M_{xy}}{M} = \frac{1}{\ln 2}. \end{aligned}$$

□

Summary

(1) Given a parameterized surface $\mathbf{r}(u,v)$

(a) Surface integral of a scalar function f:

$$\iint_{\mathbf{r}(D)} f d\sigma = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\| du dv$$

(b) Scalar surface element:

$$d\sigma = \|\mathbf{r}_u \times \mathbf{r}_v\| du dv$$

(c) Integral of a vector field:

$$\iint_{\mathbf{r}(D)} \mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, du dv = \iint_S (\mathbf{F}\cdot\mathbf{n})\, d\sigma$$

(d) Vector surface element:

$$d\boldsymbol{\sigma} = (\mathbf{r}_u \times \mathbf{r}_v)\, du dv = \mathbf{n}\, d\sigma$$

(2) When the surface is given by a graph $z = g(x,y)$

(a) Integral of a scalar f:

$$\iint_S f d\sigma = \iint_D f(x,y,g(x,y))\sqrt{(g_x)^2 + (g_y)^2 + 1}\; dx dy$$

(b) Scalar surface element:

$$d\sigma = \frac{dx\, dy}{\cos\theta} = \sqrt{(g_x)^2 + (g_y)^2 + 1}\; dx dy$$

(c) Integral of a vector field:

$$\iint_{\mathbf{S}} \mathbf{F} \cdot d\boldsymbol{\sigma} = \iint_{D} (-F_1 g_x - F_2 g_y + F_3)\, dxdy$$

(d) Vector surface element:

$$d\boldsymbol{\sigma} = \mathbf{n}\, d\sigma = (-g_x \mathbf{i} - g_y \mathbf{j} + \mathbf{k})\, dxdy$$

(3) On the sphere $x^2 + y^2 + z^2 = R^2$

(a) Scalar surface element:

$$d\sigma = R^2 \sin\phi\, d\phi d\theta$$

(b) Vector surface element:

$$d\boldsymbol{\sigma} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) R \sin\phi\, d\phi d\theta = \mathbf{r} R \sin\phi\, d\phi d\theta = \mathbf{n} R^2 \sin\phi\, d\phi d\theta$$

15.7 Stokes' Theorem

In $\mathbb{R}^2$, the vector form of Green's theorem gives the relation between the line integral of a vector field on a simple closed curve to the integral of the curl of the vector on the domain having the curve as boundary.

Stokes' theorem is the generalization of Green's theorem to the surface lying in $\mathbb{R}^3$: Consider a simple closed curve lying in $\mathbb{R}^3$ and a surface having the curve as boundary: Caution: there are many surfaces having the same curve as boundary. But as long as the vector fields are C^1 in a large region containing the curve and the surface, any surface play the same role.

Recall : the curl of $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is

$$\begin{aligned}\nabla \times \mathbf{F} = \operatorname{curl} \mathbf{F} &= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.\end{aligned}$$

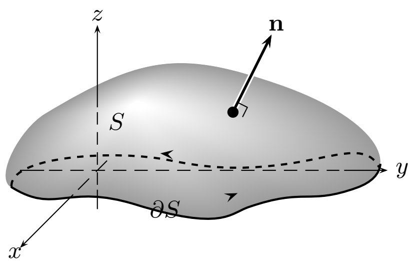


Figure 15.31: The direction of ∂S in the orientable Surface S

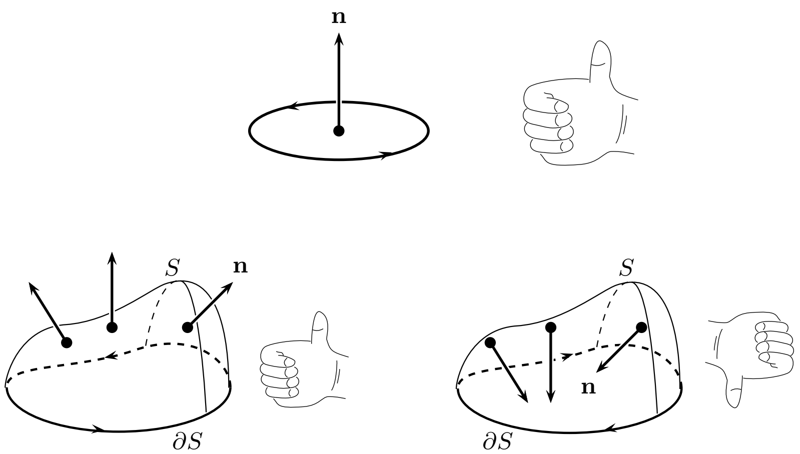


Figure 15.32: Orientation by right handed rule

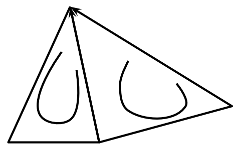
Theorem 15.7.1 (Stokes' theorem)**.** *Let S be a piecewise smooth oriented surface. Suppose the boundary ∂S consists of finitely many piecewise C^1 curve with the same orientation with S. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a C^1-vector field defined on S. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

For a 2D surface this reduces to the Green's Theorem:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA = \iint_S \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big) dxdy = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

Corollary 15.7.2. *If S_1 and S_2 are two surfaces having the same boundary,*



then

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma.$$

Proof. Polyhedral surface only!

First assume S is defined by C^1-function $z = f(x, y)$ on a two dimensional region D (a region on which Green's theorem holds). Then it can be parameterized by

$$\begin{cases} x = x \\ y = y \\ z = f(x, y), \end{cases}$$

for (x, y) in D. Recall the integral of a vector field $\mathbf{G} = G_1\,\mathbf{i} + G_2\,\mathbf{j} + G_3\,\mathbf{k}$ over S is defined by

$$\iint_S \mathbf{G} \cdot d\boldsymbol{\sigma} = \iint_D \left[G_1 \left(-\frac{\partial z}{\partial x} \right) + G_2 \left(-\frac{\partial z}{\partial y} \right) + G_3 \right] dxdy. \tag{15.25}$$

By (15.25)

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot d\boldsymbol{\sigma} &= \iint_D \Big[\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \Big) \left(-\frac{\partial z}{\partial x} \right) \\ &\quad + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \Big) \left(-\frac{\partial z}{\partial y} \right) + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \Big) \Big] dxdy. \end{aligned}$$

On the other hand

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{c}} F_1 dx + F_2 dy + F_3 dz.$$

Here $\mathbf{c} = \mathbf{F} \circ \mathbf{r}$ is a parametrization of **boundary curve** ∂S obtained from a parametrization of ∂D in positive direction. Assume ∂D has the orientation induced by $\mathbf{c}$. Then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt. \tag{15.26}$$

By the chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Substituting this into above

$$\begin{aligned}
\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \left[\left(F_1 + F_3\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(F_2 + F_3\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\right] dt \\
&= \int_{\mathbf{c}} \left(F_1 + F_3\frac{\partial z}{\partial x}\right) dx + \left(F_2 + F_3\frac{\partial z}{\partial y}\right) dy \qquad (15.27) \\
&= \int_{\partial D} \left(F_1 + F_3\frac{\partial z}{\partial x}\right) dx + \left(F_2 + F_3\frac{\partial z}{\partial y}\right) dy.
\end{aligned}$$

Applying Green's theorem to (15.27) yields

$$\iint_D \left[\left(\frac{\partial(F_2 + F_3\frac{\partial z}{\partial y})}{\partial x} - \frac{\partial(F_1 + F_3\frac{\partial z}{\partial x})}{\partial y}\right)\right] dxdy.$$

Now use chain rule keeping in mind that F_1, F_2, F_3 are functions of x, y and z, while z is again a function of x, y. (Here $\frac{\partial F_2}{\partial x}$ has to be interpreted carefully. For example, we let $G(x,y) = F_2(x, y, f(x,y))$, and $\frac{\partial F_2}{\partial x}$ is understood as $\frac{\partial G}{\partial x}$. In other words, treat x, y as independent variables, while regarding z as dependent variable.) Thus by chain rule, above integral becomes

$$\begin{aligned}
&\iint_D \left[\left(\frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F_3}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial F_3}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + F_3\frac{\partial^2 z}{\partial x \partial y}\right)\right. \\
&\left. - \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F_3}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial F_3}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + F_3\frac{\partial^2 z}{\partial x \partial y}\right)\right] dA.
\end{aligned}$$

Because mixed partials are equal, the last two integrals cancel each other and we obtain

$$\begin{aligned}
&\iint_D \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\left(-\frac{\partial z}{\partial x}\right) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\left(-\frac{\partial z}{\partial y}\right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right] dxdy \\
= &\iint_S \operatorname{curl} \mathbf{F} \cdot d\boldsymbol{\sigma}.
\end{aligned}$$

□

Example 15.7.3. Let S be smooth surface having an oriented simple closed curve C as boundary and let $\mathbf{F} = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix} = 0.$$

By Stokes' theorem,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\boldsymbol{\sigma} = 0.$$

Example 15.7.4. Verify Stoke's theorem for the following case. Let S be hemisphere $x^2+y^2+z^2 = 9, z \geq 0$ and let $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.

The boundary of S is parameterized by $\mathbf{r}(\theta) = 3\cos\theta\mathbf{i} + 3\sin\theta\mathbf{j},\ 0 \leq \theta \leq 2\pi$. First we compute $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$$\begin{aligned} d\mathbf{r} &= -3\sin\theta d\theta\mathbf{i} + 3\cos\theta d\theta\mathbf{j} \\ \mathbf{F}\cdot d\mathbf{r} &= -9\sin^2\theta d\theta - 9\cos^2\theta d\theta = -9d\theta \\ \oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} -9d\theta = -18\pi. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbf{curl\ F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\mathbf{k} \\ \mathbf{n} &= \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{3} \\ d\sigma &= \frac{|\nabla f|}{\nabla f\cdot\mathbf{k}}dA = \frac{3}{z}dA \\ \nabla\times\mathbf{F}\cdot\mathbf{n}d\boldsymbol{\sigma} &= -\frac{2z}{3}\frac{3}{z}dA = -2dA \\ \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}d\boldsymbol{\sigma} &= \iint_{x^2+y^2\leq 9} -2dA = -18\pi. \end{aligned}$$

Example 15.7.5. Compute the circulation around C in Example above using the disk of radius 3 in the xy plane centered at the origin(instead of hemisphere).

$\boxed{\textbf{sol.}}$

As before $\nabla \times \mathbf{F} = -2\mathbf{k}$ and $\mathbf{n} = \mathbf{k}$. So

$$\nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = -2\mathbf{k} \cdot \mathbf{k} dA = -2dA$$

and

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_{x^2+y^2 \le 9} -2dA = -18\pi (\textit{the same})$$

This is easier!

□

Example 15.7.6. Calculate the circulation of $\mathbf{F} = (x^2 - y)\mathbf{i} + 4z\mathbf{j} + x^2\mathbf{k}$ around the circle C where the plane $z = 2$ meets the cone $z = \sqrt{x^2 + y^2}$, counterclockwise. (In two ways)

sol. One way is to directly compute the circulation (Easy, skip it). But another way is to use Stokes' theorem on the given surface. This make things worse!!! (see book Example 4, p. 1019)

Example 5 of book 13th version. However, we can use a flat disc $z = 2$ having the same curve C as the boundary. On that disc $\mathbf{n} = \mathbf{k}$ and $\nabla \times \mathbf{F} = -4\mathbf{i} - 2x\mathbf{j} + \mathbf{k}$. $\nabla \times \mathbf{F} \cdot \mathbf{n} = 1$. So by Stokes theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_{x^2+y^2 \le 4} 1 dA = 4\pi.$$

□

Example 15.7.7. Example 6 of book. Consider a surface S formed by hyperbolic paraboloid $z = y^2 - x^2$ lying inside the cylinder of radius one around z axis and the boundary curve C. (Fig 15.33) Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$. (assume normal vector has positive $\mathbf{k}$ component on S)

sol. First we find the boundary curve C. Since it is the intersection of the hyperbolic paraboloid $z = y^2 - x^2$ with the cylinder $r = 1$, we can use

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + (\sin^2 t - \cos^2 t)\mathbf{k}$$

We calculate the circulation of $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$ around the boundary curve C.

$$\frac{d\mathbf{r}}{dt} = -\sin t\mathbf{i} + \cos t\mathbf{j} + (4 \sin t \cos t)\mathbf{k}$$

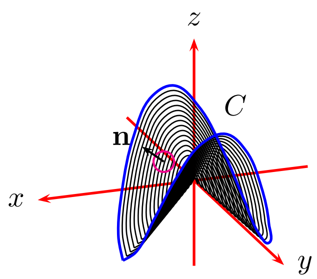


Figure 15.33: Surface $z = y^2 - x^2, x^2 + y^2 \leq 1$ for Example 15.7.7

and on the curve $\mathbf{r}$ the vector field is

$$\mathbf{F} = \sin t\mathbf{i} - \cos t\mathbf{j} + \cos^2 t\mathbf{k}$$

$$\begin{aligned}\int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt &= \int_0^{2\pi} \left(-\sin^2 t - \cos^2 t + 4\sin t \cos^3 t\right) dt \\ &= \int_0^{2\pi} \left(4\sin t \cos^3 t - 1\right) dt = -2\pi\end{aligned}$$

However, the use of Stokes' theorem for this problem make it worse, terrible!!!

□

Example 15.7.8. Verify the Stokes' theorem when $\mathbf{F} = (x^2 + y)\mathbf{i} + (x^2 + 2y)\mathbf{j} + 2z^3\mathbf{k}$ and $C : x^2 + y^2 = 4,\ z = 2$.

sol. Show that $\int_C \mathbf{F} \cdot d\mathbf{s} = -4\pi$(easy). Let S be the disk $\{(x, y, z) : x^2+y^2 = 4, z = 2\}$. If $\mathbf{n}$ is the unit normal to S, then $\mathbf{n} = \mathbf{k}$ and

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y & x^2 + 2y & 2z^3 \end{vmatrix} \\ &= (0 - 0)\mathbf{i} - (0 - 0)\mathbf{j} + (2x - 1)\mathbf{k} = (2x - 1)\mathbf{k}.\end{aligned}$$

Hence

$$\begin{aligned}\iint_S (\nabla\times\mathbf{F})\cdot d\boldsymbol{\sigma} &= \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}d\sigma\\ &= \iint_S (2x-1)\mathbf{k}\cdot\mathbf{k}d\sigma = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}(2x-1)dxdy\\ &= -2\int_{-2}^{2}\sqrt{4-y^2}dy = -4\pi = \int_C \mathbf{F}\cdot d\mathbf{s}.\end{aligned}$$

□

Example 15.7.9. Evaluate

$$\int_C -y^3dx + x^3dy - z^3dz$$

where C is the intersection of the cylinder $x^2+y^2=1$ and plane $x+y+z=1$.

sol. Let $\mathbf{F} = -y^3\mathbf{i} + x^3\mathbf{j} - z^3\mathbf{k}$. Then above integral is $\int_C \mathbf{F}\cdot d\mathbf{r}$. If we consider any reasonable surface S having C as boundary, we can use Stokes' theorem with $\operatorname{curl}\mathbf{F} = 3(x^2+y^2)\mathbf{k}$. Let us assume S is the surface defined by $x+y+z=1$, $x^2+y^2\le 1$. A parametrization of S is given by $\mathbf{r}=(u,v,1-u-v)$. We need to compute

$$d\boldsymbol{\sigma} = (\mathbf{r}_u\times\mathbf{r}_v)dudv = (\mathbf{i}-\mathbf{k})\times(\mathbf{j}-\mathbf{k})dudv = (\mathbf{i}+\mathbf{j}+\mathbf{k})dudv.$$

Hence

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_D 3(x^2+y^2)dxdy = \frac{3\pi}{2}.$$

Here the domain D is the set $\{(x,y)|x^2+y^2\le 1\}$.

□

Example 15.7.10. A surface S is defined by $z = e^{-(x^2+y^2)}$ for $z\ge 1/e$. Let

$$\mathbf{F} = (e^{y+z}-2y)\mathbf{i} + (xe^{y+z}+y)\mathbf{j} + e^{x+y}\mathbf{k}.$$

Evaluate $\iint_S \nabla\times\mathbf{F}\cdot d\boldsymbol{\sigma}$.

sol. We see

$$\nabla\times\mathbf{F} = (e^{x+y}-xe^{y+z})\mathbf{i} + (e^{y+z}-e^{x+y})\mathbf{j} + 2\mathbf{k}$$

and

$$\mathbf{N} = 2xe^{-(x^2+y^2)}\mathbf{i} + 2ye^{-(x^2+y^2)}\mathbf{j} + \mathbf{k}.$$

So direct computation of $\iint_S \nabla \times \mathbf{F} \cdot d\boldsymbol{\sigma}$ seems almost impossible. Now try to use Stoke's theorem. First parameterize the boundary by

$$x = \cos t, y = \sin t, z = 1/e.$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (e^{\sin t + 1/e} - 2\sin t, \cdots, e^{\cos t + \sin t}) \cdot (-\sin t, \cos t, 0)\, dt$$

This again is very difficult! Now think of another way. Think of another surface S' which has the same boundary as S., i.e, let S' be the unit disk $x^2 + y^2 \leq 1, z = 1/e$. Then $\mathbf{n} = \mathbf{k}$ and hence

$$\iint_S \nabla \times \mathbf{F} \cdot d\boldsymbol{\sigma} = \iint_{S'} \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_{S'} 2 d\sigma = 2\pi.$$

$\square$

Curl as circulation - paddle wheel interpretation

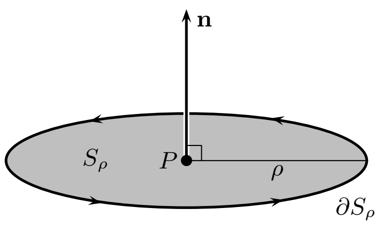


Figure 15.34: Circulation attains maximum when $S_\rho \perp \mathbf{n}$

Suppose $\mathbf{F}$ represent the velocity of a fluid. Consider a disk S_ρ centered at P with radius ρ having normal vector $\mathbf{n}$ (fig 15.34). Then the circulation around the disk S_ρ is

$$\int_{\partial S_\rho} \mathbf{F} \cdot d\mathbf{r}.$$

By Stokes' theorem,

$$\int_{\partial S_\rho} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_\rho} (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma. \tag{15.28}$$

Here ∂S_ρ has the orientation according to $\mathbf{n}$. We have by MVT

$$\iint_{S_\rho} \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = [\nabla \times \mathbf{F}(Q) \cdot \mathbf{n}(Q)]\pi\rho^2$$

for some point Q in S_ρ. Hence dividing equation (15.28)by $\pi\rho^2$ we have

$$\begin{aligned}\lim_{\rho\to 0} \frac{1}{\pi\rho^2} \int_{\partial S_\rho} \mathbf{F} \cdot d\mathbf{r} &= \lim_{\rho\to 0} \frac{1}{\pi\rho^2} \iint_{S_\rho} (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma \\ &= \lim_{\rho\to 0} (\nabla \times \mathbf{F}) \cdot \mathbf{n}(Q) \\ &= (\nabla \times \mathbf{F}) \cdot \mathbf{n}(P).\end{aligned}$$

The circulation will attain its maximum value when the normal vector to the plane is parallel to $\nabla \times \mathbf{F}$. Thus the normal component of the curl at P is the circulation density.(Note that the circulation density is, in the limit as $\rho \to 0$, a 2D concept, while the curl of a vector field is a 3 D concept)

Example 15.7.11. A fluid rotates around the z-axis with velocity $\mathbf{F} = \omega(-y\mathbf{i} + x\mathbf{j})$, where $\omega > 0$ is the angular velocity. Find $\nabla \times \mathbf{F}$ and relate it to the circulation density.

sol.

$$\nabla \times \mathbf{F} = 2\omega\mathbf{k}.$$

Let S be the disk of radius ρ at the origin in the plane normal to $\nabla \times \mathbf{F}$ (in this case the xy plane). By Stokes' theorem, we have

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = \iint_S 2\omega\mathbf{k} \cdot \mathbf{k} dxdy = 2\omega\pi\rho^2.$$

Thus

$$2\omega = \frac{1}{\pi\rho^2} \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \nabla \times \mathbf{F} \cdot \mathbf{k}.$$

This says that the circulation density is the normal component of $\nabla \times \mathbf{F}$ (i.e., $\mathbf{n} = \mathbf{k}$)

$\square$

Example 15.7.12. Use Stokes' theorem to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F} = xz\mathbf{i} + xy\mathbf{j} + 3xz\mathbf{k}$, and C is the boundary of S, which is the plane $2x + y + z = 2$ portion in the first octant.

sol. The plane is the level surface of $f(x, y, z) = 2x + y + z = 2$. Thus the unit normal vector is

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2\mathbf{i} + \mathbf{j} + \mathbf{k}}{|2\mathbf{i} + \mathbf{j} + \mathbf{k}|} = \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\nabla \times \mathbf{F} = 2\omega\mathbf{k}.$$

By Stokes' theorem, we have

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = \iint_S 2\omega\mathbf{k} \cdot \mathbf{k} dxdy = 2\omega\pi\rho^2.$$

Thus

$$2\omega = \frac{1}{\pi\rho^2} \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \nabla \times \mathbf{F} \cdot \mathbf{k}.$$

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xy & 3xz \end{vmatrix} = (x - 3z)\mathbf{j} + y\mathbf{k} \\
&= (x - 3(2 - 2x - y))\mathbf{j} + y\mathbf{k} = (7x + 3y - 6)\mathbf{j} + y\mathbf{k} \\
\nabla \times \mathbf{F} \cdot \mathbf{n} &= \frac{1}{\sqrt{6}}(7x + 4y - 6)
\end{aligned}$$

The surface area element is

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} dxdy = \sqrt{6} dxdy$$

Thus the circulation is

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S (\nabla\times\mathbf{F})\cdot \mathbf{n}d\sigma\\ &= \int_0^1\int_0^{2-2x} \frac{1}{\sqrt{6}}(7x+4y-6)\sqrt{6}dxdy\\ &= \int_0^1\int_0^{2-2x}(7x+4y-6)dxdy = -1.\end{aligned}$$

□

Example 15.7.13. Let S be the elliptic paraboloid $z = x^2+4y^2$ lying beneath the plane $z = 1$. Define the normal vector $\mathbf{n}$ pointing upward.(i.e, having positive $\mathbf{k}$ component) Find the flux of $\nabla\times\mathbf{F}$ across S when $\mathbf{F} = y\mathbf{i}-xz\mathbf{j}+xz^2\mathbf{k}$.

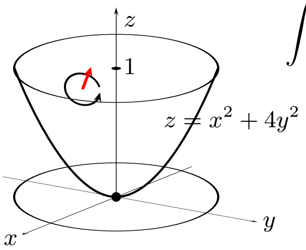
sol. Let $C : x^2 + 4y^2 = 1$ be the boundary of S with correct orientation. We parameterize it by

$$\mathbf{r}(t) = \cos t\mathbf{i} + \frac{1}{2}\sin t\mathbf{j} + \mathbf{k}$$

Then

$$\mathbf{F}(\mathbf{r}(t)) = \frac{1}{2}\sin t\mathbf{i} - \cos t\mathbf{j} + \cos t\mathbf{k}, \qquad \frac{d\mathbf{r}(t)}{dt} = -\sin t\mathbf{i} + \frac{1}{2}\cos t\mathbf{j}.$$

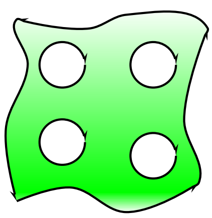
Thus



$$\begin{aligned}\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}d\sigma &= \int_C \mathbf{F}\cdot d\mathbf{r}\\ &= -\frac{1}{2}\int_0^{2\pi}(\sin^2 t + \cos^2 t)dt\\ &= -\pi.\end{aligned}$$

□

Stokes' theorem for surfaces with holes



Important identity

$$\operatorname{curl}\operatorname{grad} f = \nabla \times \nabla f = 0 \tag{15.29}$$

$$\begin{aligned}\nabla \times \nabla f &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = (f_{zy} - f_{yz})\mathbf{i} + (f_{xz} - f_{zx})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k} \\ &= 0\end{aligned}$$

Conservative field and Stokes' theorem

Recall :$\mathbf{F}$ is said to be conservative if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve. Equivalently, by Stokes' theorem, $\mathbf{F}$ is conservative if $\nabla \times \mathbf{F} = 0$ in a simply connected region. Note this fact is consistent with the Stokes' theorem:

$$\boxed{\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma.}$$

15.8 Divergence Theorem

We define the divergence of a vector field $\mathbf{F}$ as

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

Physical meaning of divergence: Expansion or compression of a material.

Theorem 15.8.1. *[**Gauss' Divergence Theorem**] Let Ω be an elementary region in $\mathbb{R}^3$ and $\partial\Omega$ consists of finitely many oriented piecewise smooth closed surfaces. Let $\mathbf{F}$ be a $\mathcal{C}^1$-vector field on a region containing Ω. Then*

$$\iint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_{\Omega} \operatorname{div} \mathbf{F} dV.$$

The flux of a vector field $\mathbf{F}$ across Ω is equal to the integral of $\operatorname{div} \mathbf{F}$ in Ω.

Example 15.8.2. S is the unit sphere $x^2+y^2+z^2 = 1$ and $\mathbf{F} = 2x\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$. Find $\iint_S \mathbf{F} \cdot \mathbf{n} dS$.

sol. Let Ω be the region inside S. By Gauss theorem, it holds that

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_\Omega \operatorname{div} \mathbf{F} dV.$$

Since $\operatorname{div} \mathbf{F} = \nabla \cdot (2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) = 2(1 + y + z)$, the rhs is

$$2\iiint_\Omega (1 + y + z) dV = 2\iiint_\Omega 1 dV + 2\iiint_\Omega y dV + 2\iiint_\Omega z dV.$$

By symmetry, we have

$$\iiint_\Omega y dV = \iiint_\Omega z dV = 0.$$

Hence

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = 2\iiint_\Omega (1 + y + z) dV = 2\iiint_\Omega 1 dV = \frac{8}{3}\pi.$$

□

Example 15.8.3. Find the flux of $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ through the box cut from the first octant by the planes $x = 1, y = 1, z = 1$.

sol. Let Ω be the region inside S. By Gauss theorem, it holds that

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_\Omega \operatorname{div} \mathbf{F} dV.$$

Since $\operatorname{div} \mathbf{F} = \nabla \cdot (xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}) = x + y + z$, the rhs is

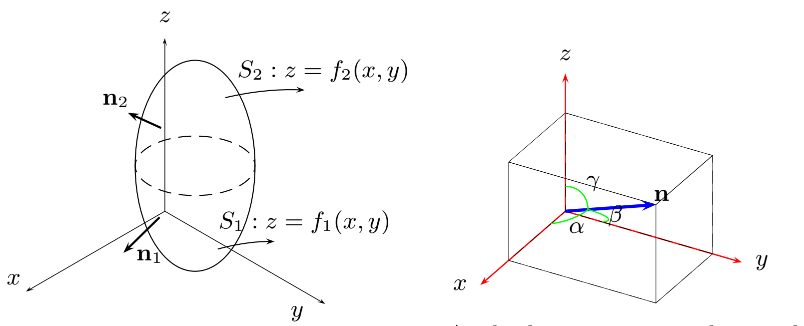
$$\iiint_\Omega (x + y + z) dV = \int_0^1 \int_0^1 \int_0^1 (x + y + z) dx dy dz = \frac{3}{2}.$$

□

Proof. (of Divergence Theorem) Suppose Ω is an elementary region of type 4 (a convex region like a ball bounded by graphs of two functions:

$$\begin{aligned} S_1 &: z = f_1(x, y), \quad (x, y) \in R_{xy} \\ S_2 &: z = f_2(x, y), \quad (x, y) \in R_{xy}, \end{aligned}$$

with $f_1(x, y) \le f_2(x, y)$, see fig 15.35. The unit normal vector $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} +$



Angles between axes and normal

Figure 15.35: Region of type 1

$n_3\mathbf{k}$ satisfies

$$\begin{aligned} n_1 &= \mathbf{n} \cdot \mathbf{i} = \cos\alpha \\ n_2 &= \mathbf{n} \cdot \mathbf{j} = \cos\beta \\ n_3 &= \mathbf{n} \cdot \mathbf{k} = \cos\gamma. \end{aligned}$$

Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$. Then the surface integral is

$$\begin{aligned} \iint_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) d\sigma &= \iint_{\partial\Omega} (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) \cdot \mathbf{n} d\sigma \\ &= \iint_{\partial\Omega} M \cos\alpha d\sigma + \iint_{\partial\Omega} N \cos\beta d\sigma + \iint_{\partial\Omega} P \cos\gamma d\sigma, \end{aligned}$$

On the other hand,

$$\iiint_{\Omega} \operatorname{div} \mathbf{F} dV = \iiint_{\Omega} \frac{\partial M}{\partial x} dV + \iiint_{\Omega} \frac{\partial N}{\partial y} dV + \iiint_{\Omega} \frac{\partial P}{\partial z} dV.$$

If we show the following the proof will be complete.

$$\iint_{\partial\Omega} P\mathbf{i} \cdot \mathbf{n} d\sigma = \iint_{\partial\Omega} P \cos\alpha d\sigma = \iiint_{\Omega} \frac{\partial P}{\partial x} dV, \tag{15.30}$$

$$\iint_{\partial\Omega} Q\mathbf{j} \cdot \mathbf{n} d\sigma = \iint_{\partial\Omega} P \cos\beta d\sigma = \iiint_{\Omega} \frac{\partial Q}{\partial y} dV, \tag{15.31}$$

$$\iint_{\partial\Omega} R\mathbf{k} \cdot \mathbf{n} d\sigma = \iint_{\partial\Omega} P \cos\gamma d\sigma = \iiint_{\Omega} \frac{\partial R}{\partial z} dV. \tag{15.32}$$

First we shall prove (15.32). Since

$$d\sigma = \frac{dx\,dy}{\cos\gamma},$$

we have

$$\begin{aligned}
\iint_{\partial\Omega} P\cos\gamma d\sigma &= \iint_{S_2} P\cos\gamma d\sigma + \iint_{S_1} P\cos\gamma d\sigma && (15.33)\\
&= \iint_D [R(x,y,f_2(x,y)) - R(x,y,f_1(x,y))]dxdy. && (15.34)\\
&= \iint_D \left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial R}{\partial z}dz\right) dxdy = \iiint_\Omega \frac{\partial R}{\partial z}dV. && (15.35)
\end{aligned}$$

The identities (15.30) and (15.31) can be similarly shown. □

Theorem 15.8.4. *[**Divergence of curl**] Let* $\mathbf{F}$ *be a* $\mathcal{C}^2$ *vector field defined on a region containing* Ω*. Then*

$$div\,(\mathrm{curl}\mathbf{F}) = \nabla\cdot(\nabla\times\mathbf{F}) = 0.$$

Proof.

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})\mathbf{i} + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})\mathbf{j} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\mathbf{k}$$

$$\mathrm{div}\,\mathbf{F} = 0$$

□

Example 15.8.5. Show Gauss' theorem holds for $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in $\Omega : x^2+y^2+z^2 \le a^2$.

sol. First compute $\mathrm{div}\,\mathbf{F} = \nabla\cdot\mathbf{F}$,

$$\mathrm{div}\,\mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

So

$$\iiint_\Omega (\mathrm{div}\,\mathbf{F})dV = \iiint_\Omega 3\,dV = 3\Big(\frac{4}{3}\pi a^3\Big) = 4\pi a^3.$$

To compute the surface integral, we need to find the unit normal $\mathbf{n}$ on $\partial\Omega$.

Since $\partial\Omega$ is the level set of $f(x,y,z) = x^2+y^2+z^2-a^2$, we see the unit normal vector to $\partial\Omega$ is

$$\mathbf{n} = \frac{\nabla f}{||\nabla f||} = \frac{2(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{\sqrt{4(x^2+y^2+z^2)}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}.$$

So when $(x,y,z) \in \partial\Omega$,

$$\mathbf{F}\cdot\mathbf{n} = \frac{x^2+y^2+z^2}{a} = \frac{a^2}{a} = a$$

and

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}d\sigma = \iint_{\partial\Omega} a\,d\sigma = a(4\pi a^2) = 4\pi a^3.$$

Hence

$$\iiint_{\Omega} (\operatorname{div}\mathbf{F})dV = 4\pi a^3 = \iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}d\sigma.$$

and Gauss' theorem holds.

□

Example 15.8.6. Let Ω be the region given by $x^2+y^2+z^2 \le 1$. Find $\iint_{\partial\Omega}(x^2+4y-5z)d\sigma$ by Gauss' theorem.

$\boxed{\textbf{sol.}}$ To use Gauss' theorem, we need a vector field $\mathbf{F} = F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$ such that $\mathbf{F}\cdot\mathbf{n} = x^2+4y-5z$. Since the unit normal vector is $\mathbf{n} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, one such obvious choice is $\mathbf{F} = x\mathbf{i}+4\mathbf{j}-5\mathbf{k}$. Hence we have $\operatorname{div}\mathbf{F} = 1+0+(-0) = 1$. Now by Gauss theorem

$$\begin{aligned}\iint_{\partial\Omega}(x^2+4y-5z)d\sigma &= \iint_{\partial\Omega}(x\mathbf{i}+4\mathbf{j}-5\mathbf{k})\cdot\mathbf{n}d\sigma \\ &= \iint_{\partial\Omega}\mathbf{F}\cdot\mathbf{n}d\sigma = \iiint_{\Omega}\operatorname{div}\mathbf{F}dV \\ &= \iiint_{\Omega} 1\; dV = \frac{4}{3}\pi.\end{aligned}$$

□

Example 15.8.7. Let Ω be the region satisfying $0 < b^2 \le x^2+y^2+z^2 \le a^2$. Find the flux of the vector field $\mathbf{F} = (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/\rho^3$, $\rho = \sqrt{x^2+y^2+z^2}$ across the boundary of Ω.

$\boxed{\textbf{sol.}}$ On the boundary of Ω, $\mathbf{n} = \pm(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/\rho$. Hence $\mathbf{F}\cdot\mathbf{n} = \pm(x\mathbf{i}+$

$y\mathbf{j} + z\mathbf{k}$),

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}d\sigma = \iint_{S_a} \mathbf{F}\cdot\mathbf{n}d\sigma - \iint_{S_b} \mathbf{F}\cdot\mathbf{n}d\sigma$$

$$\iint_{S_a} \mathbf{F}\cdot\mathbf{n}d\sigma = \iint_{\rho=a} \frac{1}{\rho^2}d\sigma = 4\pi.$$

Since this value is independent of a,

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}d\sigma = 4\pi - 4\pi = 0.$$

To use Gauss' theorem, we compute that $\nabla\cdot\mathbf{F} = 0$. Hence Now by Gauss theorem

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}d\sigma = \iiint_{\Omega} \operatorname{div}\mathbf{F}dV = 0.$$

□

Generalizing Gauss' divergence theorem

Divergence theorem holds for more general regions. The idea is the break the region into subregions of type 4.

Divergence as flux per unit Volume

As we have seen before that $\operatorname{div}\mathbf{F}(P)$ is the *rate of change of total flux* at P per unite volume. Let Ω_ρ be a ball of radius ρ center at P. Then for some Q in Ω_ρ,

$$\iint_{\partial\Omega_\rho} \mathbf{F}\cdot\mathbf{n}d\sigma = \iiint_{\Omega_\rho} \operatorname{div}\mathbf{F}dV = \operatorname{div}\mathbf{F}(Q)\cdot\operatorname{Vol}(\Omega_\rho).$$

Dividing by the volume we get

$$\operatorname{div}\mathbf{F}(Q) = \frac{1}{\operatorname{Vol}(\Omega_\rho)}\iint_{\partial\Omega_\rho} \mathbf{F}\cdot\mathbf{n}d\sigma. \tag{15.36}$$

Taking the limit, we see

$$\lim_{\rho\to 0}\frac{1}{\operatorname{Vol}(\Omega_\rho)}\iint_{\partial\Omega_\rho} \mathbf{F}\cdot\mathbf{n}d\sigma = \operatorname{div}\mathbf{F}(P). \tag{15.37}$$

Now we can give a physical interpretation: If $\mathbf{F}$ is the velocity of a fluid, then

$$\boxed{\text{div}\,\mathbf{F}(P) \text{ is the rate at which the fluid flows out per unit volume.}}$$

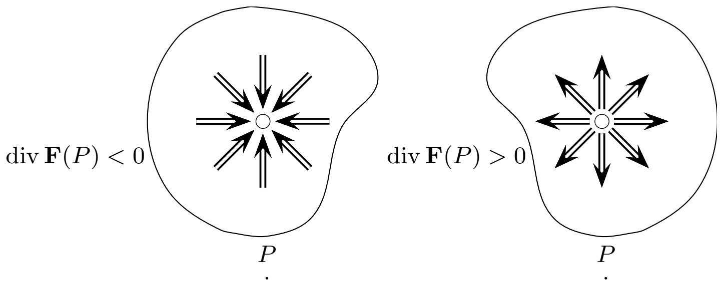


Figure 15.36:

If div $\mathbf{F}(P) > 0$, we say P is a **source** and if div $\mathbf{F}(P) < 0$, it is called **sink** of $\mathbf{F}$(fig 15.36).

If div $\mathbf{F} = 0$ then by Gauss theorem, the total flux of $\mathbf{F}$ through any closed surface S is $\iint_S \mathbf{F} \cdot d\boldsymbol{\sigma}$, which is zero. Thus we call this vector field **incompressible**.

Example 15.8.8. Find $\iint_S \mathbf{f} \cdot d\boldsymbol{\sigma}$, where $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$ and S is the surface of the the cylindrical region $x^2 + y^2 = 1$ bounded by the planes $z = 1$ and $z = -1$.

sol. Let W denote the solid region given above. By divergence theorem,

$$\begin{aligned}
\iiint_W \text{div}\,\mathbf{F}\,dV &= \iiint_W (x^2 + y^2)dxdydz \\
&= \int_{-1}^{1} \left(\iint_{x^2+y^2\le 1} (x^2 + y^2)dxdy \right) dz \\
&= 2 \iint_{x^2+y^2\le 1} (x^2 + y^2)dxdy.
\end{aligned}$$

Now by polar coordinate,

$$2 \iint_{x^2+y^2\le 1} (x^2 + y^2)dxdy = 2 \int_0^{2\pi} \int_0^1 r^3 dr d\theta = \pi.$$

□

Gauss' Law

Now apply Gauss' theorem to a region with a hole and get an important result in physics:

The electric field created by a point charge q at the origin is

$$\mathbf{E}(x,y,z) = \frac{q}{4\pi\epsilon_0}\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{r^3} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}, \; r = \sqrt{x^2+y^2+z^2}$$

Theorem 15.8.9. ***(Gauss' Law)*** *Let M be a region in $\mathbb{R}^3$ and $O \notin \partial M$. Then*

$$\iint_{\partial M} \mathbf{E}\cdot\mathbf{n}d\sigma = \frac{q}{4\pi\epsilon_0}\iint_{\partial M}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}d\sigma = \begin{cases} 0 & \text{if } O \notin M\ , \\ \frac{q}{\epsilon_0} & \text{if } O \in M. \end{cases}$$

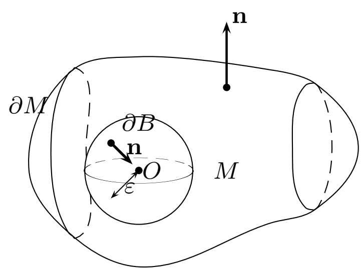


Figure 15.37: Unit outward normal vector $\mathbf{n}$ to M and Gauss' Law

Proof. First suppose $O \notin M$. Then $\mathbf{r}/r^3$ is a C^1-vector field on M and ∂M. One can easily show $\nabla\cdot(\mathbf{r}/r^3) = 0$ for $r \neq 0$. Hence

$$\iint_{\partial M}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}d\sigma = \iiint_M \nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right)dV = 0.$$

Thus we have the result.

Next, if $O \in M$, $\mathbf{r}/r^3$ is not continuous on M. Then we remove small ball B of radius ε near O(fig 15.37). Let W be the region $M\backslash B$. Then the boundary of W is $S = \partial B \cup \partial M$, where the normal vector to B is opposite to the usual direction. Again we see in $\nabla\cdot(\mathbf{r}/r^3) = 0$ in W. Hence by Gauss theorem

$$\iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}d\sigma = \iiint_W \nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right)dV = 0.$$

Since

$$\iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}d\sigma = \iint_{\partial M} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}d\sigma + \iint_{\partial B} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}d\sigma,$$

we have

$$\iint_{\partial M} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}d\sigma = -\iint_{\partial B} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}d\sigma.$$

Now on ∂B(a sphere of radius ε), we know $\mathbf{n} = -\mathbf{r}/r$ and $r = \varepsilon$. Hence

$$-\iint_{\partial B} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}d\sigma = \iint_{\partial B} \frac{\varepsilon^2}{\varepsilon^4}d\sigma = \frac{1}{\varepsilon^2}\iint_{\partial B} d\sigma.$$

Since $\iint_{\partial B} d\sigma = 4\pi\varepsilon^2$, we have $\iint_{\partial M} \mathbf{r}\cdot\mathbf{n}/r^3 d\sigma = 4\pi$. □

Several versions of Green's theorem:

Tangential form	$\oint_C \mathbf{F}\cdot\mathbf{T}ds$	$=$	$\iint_R \nabla\times\mathbf{F}\cdot\mathbf{k}dA$
Stokes' theorem	$\oint_{\partial S} \mathbf{F}\cdot\mathbf{T}ds$	$=$	$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}d\sigma$
Normal form	$\oint_C \mathbf{F}\cdot\mathbf{n}ds$	$=$	$\iint_R \nabla\cdot\mathbf{F}dA$
Divergcenc theorem	$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}d\sigma$	$=$	$\iiint_\Omega \nabla\cdot\mathbf{F}dV$